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Journal of Research of the National Bureau of Standards Vol. 57, No. 5, November 1956 Research Paper 2716

Efficiency of 4π–Crystal–Scintillation Counting: 1. Experimental Technique and Results

C. C. Smith, H. H. Seliger, and J. Steyn [1]

The technique of 4π-crystal-scintillation counting has been applied to the standardization of beta-emitting nuclides. Phototubes viewing an anthracene sandwiched source at 180° have been used with high-gain nonoverloading amplifiers in addition and in coincidence circuits to obtain high detection efficiencies. A precise shorted delay-line coincidence analyzer is described. The results of measurements made with P^{32}, Co^{60}, Sr^{90}-Y^{90}, I^{131}, and Tl^{204} are compared with those obtained by $4\pi\beta$-gas-proportional counting.

1. Introduction

The method of $4\pi\beta$-gas-proportional counting has been established for some 6 years now for the primary standardization of beta-emitting nuclides at the National Bureau of Standards [1, 2, 3].[2] The method of 4π-liquid-scintillation counting has also recently been investigated at the Royal Cancer Hospital in London [4]. It was therefore considered to be of interest to initiate experiments to determine the efficiency of 4π-crystal-scintillation counting and its applicability to primary beta standardization.

These measurements have now been concluded and show that the method of 4π-crystal-scintillation counting, although not as generally applicable as $4\pi\beta$-gas-proportional counting, can be used for medium and high-energy beta-emitting nuclides. It is the purpose of this paper to report briefly the experimental results obtained for the benefit of other interested workers in the field.

2. Experimental Procedure

The crystal scintillation counters used in these experiments consisted of sandwiches formed by two transparent anthracene crystals each approximately 1 in. in diameter and ranging from ⅛ to ¼ in. in thickness. Two Dumont 6292 multiplier phototubes, in contact with the backfaces of the anthracene crystals, were used with high-gain nonoverloading amplifiers to detect light pulses occurring within the crystals. One of the experimental arrangements is shown in figure 1. The phototubes were inside a light-tight refrigerator, with the lowered temperatures being used to reduce the random thermal tube noise. The entire sandwich is surrounded by an aluminum cylinder for complete light collection. In this connection it was found that a 2-in. diameter cylinder gave slightly better light collection than one of only slightly larger diameter than that of the crystals, presumably because the former enabled reflected light to reach the phototube directly without being retransmitted through the crystals or scattered by them.

Negative high voltage was supplied to the phototubes so that their anodes could be connected directly to the cathode-follower input grids. Voltages as high as −1,400 v were used and electronic amplifier gains of 40,000 were, in some cases, found necessary.

The radioactive material was generally introduced into the sandwich by depositing an aliquot of the active solution directly onto the top face of the lower crystal. The deposit was allowed to dry and then covered by the second crystal to form the sandwich. Where special techniques were used to precipitate the active material the sources were prepared on thin Formvar-polystyrene laminated films (10μg/cm^2) and then placed on the crystal face.

The pulses from the phototubes arising from the light pulses in the crystal sandwich were counted in two ways, namely in addition and in coincidence. In the former method the separate phototube-cathode-follower output pulses were added together and then fed into a single amplifier and counted. It was found that a single tube and hemispherical reflector arrangement was not as efficient as the two-tube addition method. In the coincidence method the output pulses were fed into separate amplifiers, thence into a coincidence analyzer of short (1μsec) resolving time and the coincidences counted.

Each of these methods, two-tube addition and coincidence, has advantages and disadvantages. In the coincidence method the random noise pulses originating in each phototube will not be counted except for those few accidental coincidences arising during the finite resolving time of the coincidence circuit. However, it is necessary that a true light pulse originating inside the sandwich be "seen" by *both* phototubes. This stringent requirement is absent in the addition method so that the sensitivity to very small light pulses is higher by at least a factor of 2 as compared with the coincidence method. On the other hand, the addition method suffers from the limitation of excessive phototube noise at the high tube gains sometimes required. Because the phototube noise is usually not reproducible to better than ±5 to 10 percent the uncertainty in the knowledge of the true disintegration rate can become

[1] Guest worker from the National Physical Laboratory, Pretoria, Union o South Africa, from January to June 1953.

[2] Figures in brackets indicate the literature references at the end of this paper.

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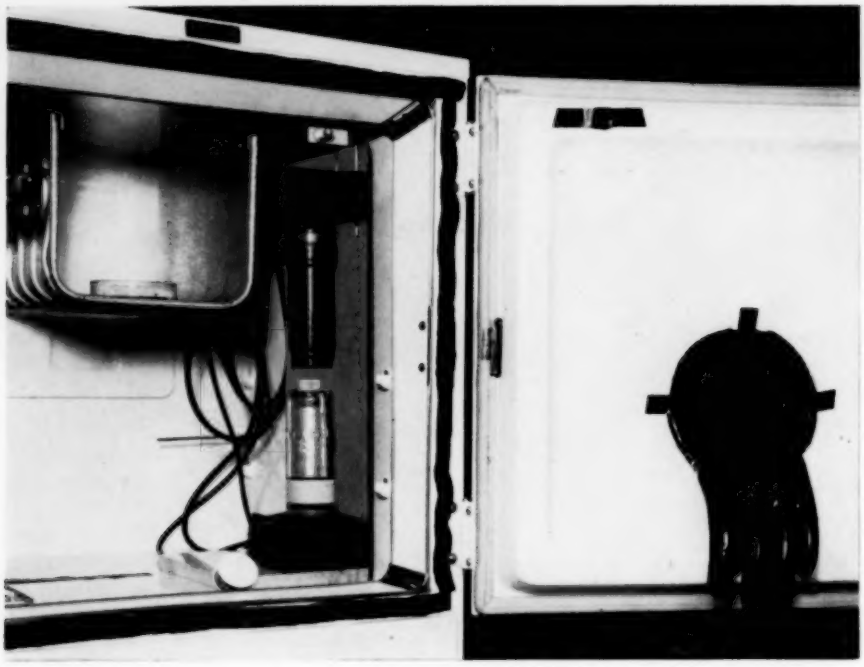

FIGURE 1. *Anthracene crystal sandwich and multiplier phototubes assembly in refrigerator.*

appreciable. A noise counting rate of 2,000 counts per second in each channel would, in the addition method, give rise to approximately 4,000 counts per second to be subtracted from the total observed counting rate. However, in the coincidence method, with a resolving time of 1 μsec, the increase in coincidence counting-rate observed would be only 4 counts per second. Because this number is a small fraction of the "true" counting rate any uncertainty in the knowledge of the exact noise background will be reduced in this ratio.

The coincidence method presupposes that at least one photoelectron is emitted from *each* photocathode in order that there be a finite probability of detecting a coincidence. Assuming a 5 percent photocathode photoelectric efficiency and a geometrical light-collecting efficiency of 100 percent, one would require a minimum of 40 photons (20 photons to each phototube) to give rise to a coincidence count. However, in order to insure that *both* photocathodes receive at least 20 photons with a 99 percent probability it would require a minimum of 68 initial photons. With the further assumption of a quantum efficiency of 65 ev/photon for anthracene [5] the effective beta-ray cutoff energy would correspond to 4.4 kev. In the addition method, however, either phototube would have a threshold of 20 photons. This requirement is not as stringent as in the coincidence method and the effective cutoff energy would be 2.2 kev. In the $4\pi\beta$-gas-proportional counting method the effective energy cutoff is estimated to be only 0.2 kev.

For high-energy beta emitters such as P^{32} and Sr^{90}-Y^{90} the preponderance of the emitted beta rays have energies greater than 2.2 kev. Further, the light pulses are rather large so that the phototube gain need not be high. Thus the noise is low and the addition method is satisfactory. However, for the lower-energy beta emitters the phototube gain must be increased, increasing the random tube noise and necessitating the use of the coincidence method.

A precision shorted delay-line coincidence analyzer was designed for the coincidence measurements and is shown schematically in figure 2. The unit accepts positive pulses from the nonoverloading amplifiers, shapes them and feeds the shaped pulses into a biased mixing stage. The rise and decay times of the square-shaped pulses are 0.06 μsec. The resolving time is variable by means of a shorted delay-line and the resolving time is independent of counting rate up to 5,000 counts per second in each channel. The single channel outputs record only those pulses that have been accepted by the input discriminator and which are on their way to the coincidence mixing stage.

3. Results and Discussion

A good fraction of the data was taken at room temperature. Operation at temperatures below 0° C considerably improved the precision of the tube-noise measurements and permitted the use of higher phototube voltages than was previously feasible. To illustrate the improvement due to lowered temperatures and consequent lowered tube noise, as well as the difference between the coincidence method and the addition method, several relative efficiency curves have been reproduced in figure 3. The efficiencies relative to $4\pi\beta$-gas-proportional counting are plotted as functions of phototube voltage for both the addition and the coincidence

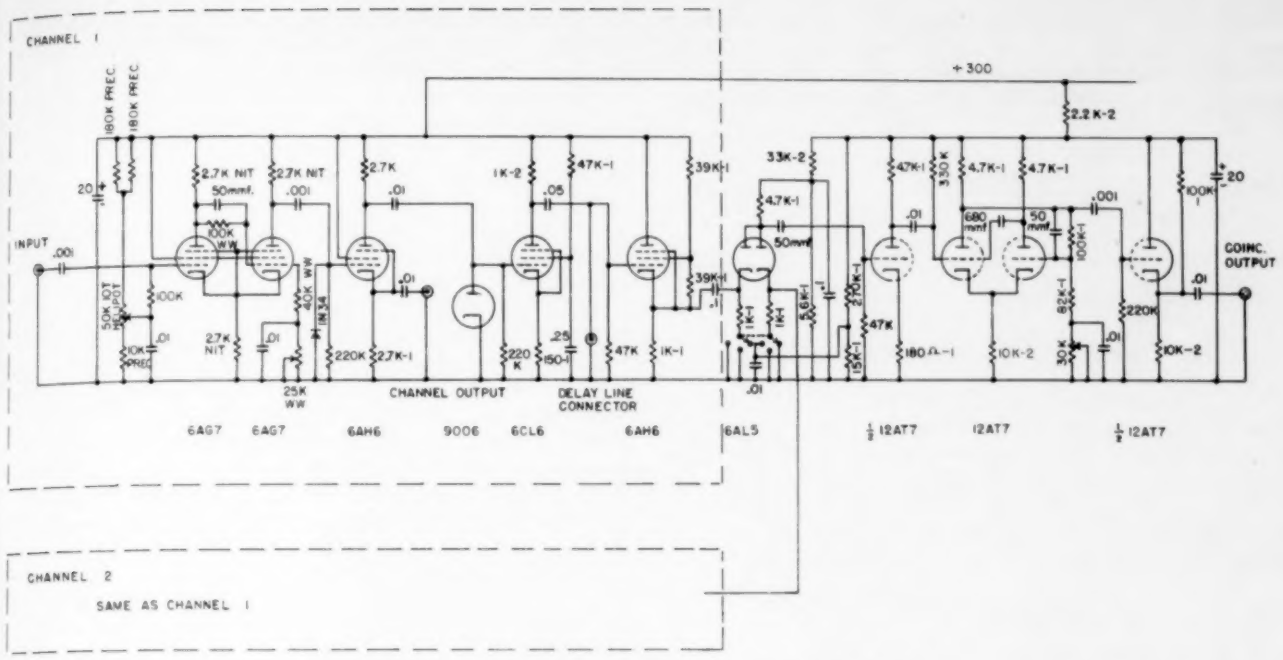

FIGURE 2. *Shorted delay-line coincidence-analyzer circuit.*

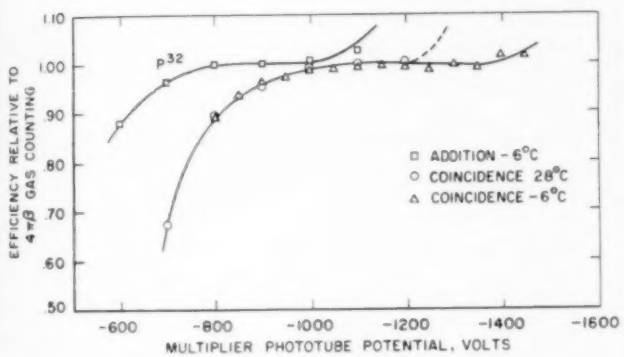

FIGURE 3. *Relative efficiency of 4π-crystal-scintillation counting to $4\pi\beta$-gas-proportional counting.*

methods at 28° C and at −6° C. In each case the usable plateau is considerably lengthened at the lowered temperature. These data were obtained for a high-energy beta emitter, namely P^{32}. For low-energy beta emitters the plateau shrinks to a very small region of phototube voltage and the improvement due to lowered temperatures is even more striking. For example, Tl^{204} ($E_{\beta max}=0.67$ Mev) exhibited a plateau only at the lowered temperature for the addition method. At room temperature the thermal-noise background at the phototube gains required was so large and erratic as to completely mask any plateau. Because of its better noise-discrimination characteristics the coincidence method was used for all of the lower-energy beta emitters, at room temperatures and at the lowered temperatures. It is particularly in these cases that the corrections for dead-time and coincidence resolving time become important. In a separate paper [6] both approximate and rigorous derivations for these counting losses are given and their range of applicability is discussed. Because it is shown that the approximate derivation can be employed under the present experimental conditions, only these corrections will be summarized here.

In channel I, consisting of phototube I, amplifier I and discriminator and gate I, a counting rate $N(\mathrm{I})$ is observed with

$$N(\mathrm{I})=N_n'+N_{te}. \tag{1}$$

N_n' is the observed thermal tube noise of phototube I and N_{te} is the observed "true" counting rate due to the source. Similarly in channel II for phototube II viewing the opposite side of the sandwich

$$N(\mathrm{II})=N_n''+N_{te}. \tag{2}$$

The dead-time losses are due to the introduction of an electronic gate τ_g ($\tau_{g\mathrm{I}}=\tau_{g\mathrm{II}}$) whose duration is in all cases longer than the resolving time of the non-overloading amplifier. The use of this fixed gate insures a precise correction for dead-time as the amplifier resolving time is dependent upon the degree of overload or gain. To a first approximation the fractional dead-time loss due to noise and "true" counts in both channels is given by

$$L=\tau_g\,(N(\mathrm{I})+N(\mathrm{II})-N_{te}+N_n'N_n''\tau_g), \tag{3}$$

remembering that N_{te} occurs in *both* channels simultaneously and should be corrected for only *once*. Again to a first approximation the observed coincidence rate N_c is related to N_{te} by

$$N_c=N_{te}+2\tau_c N_n'N_n'', \tag{4}$$

where τ_e is the resolving time of the coincidence analyzer; the last term in eq (4) being the accidental coincidence rate due to random noise. From eq (3) and (4) one obtains the total disintegration rate, N_t, as

$$N_t = \frac{N_c - 2\tau_c N_n' N_n''}{1 - \tau_g[N(\mathrm{I}) + N(\mathrm{II}) - (N_c - 2\tau_c N_n' N_n'') + N_n' N_n'' \tau_g]}. \quad (5)$$

In a blank experiment, i. e., no source present,

$$(N_c)_b = (N_{tc})_b + 2\tau_c (N_n' N_n'')_b \quad (6)$$

$(N_{tc})_b$ arises from cosmic rays, possible contamination of the crystals, and fluorescence in the glass of the phototubes. In actual practice $(N_{tc})_b \ll (N(I,\mathrm{II}))_b$ so that with very little error N_n' in eq (5) and (6) can be set equal to $(N(\mathrm{I}))_b$. Likewise $N'' \simeq (N(\mathrm{II}))_b$. The disintegration rate of the source N_0 is then given to within a few tenths of a percent by

$$N_0 = N_t - (N_{tc})_b \quad (7)$$

Figure 4 shows the extent of these corrections in the cases of Tl^{204} and Co^{60}. The dashed curves represent the observed values of $N_c - (N_c)_b$ as functions of phototube voltage for both isotopes at 28° C and at −6° C. The solid curves, representing these data corrected by means of eq (5), (6), and (7), give N_0, the actual source disintegration rate. Due to the high noise counting rates at room temperature the corrections required were appreciably higher than those for the lower temperature, making it impracticable to continue any measurements at room temperature past −1,200 v.

Table 1 gives a summary of data obtained by means of the 4π-crystal-scintillation technique for the nuclides P^{32}, Co^{60}, $Sr^{90}-Y^{90}$, I^{131}, and Tl^{204}, compared with measurements made in the $4\pi\beta$-gas-proportional counter.

This comparison points up a serious limitation of the 4π-crystal technique for low-energy beta emitters. Because the minimum pulse observable is that caused by a single photoelectron being emitted from the photocathode there is a definite energy cutoff that will depend upon (a) the intrinsic photon efficiency of the crystal, (b) the transmission of the crystal to its own light, (c) the external light-collecting geometry, and (d) the photoelectric efficiency of the photocathode. The variations in observed efficiency for Co^{60} shown in table 1 are probably due to combinations of these factors.

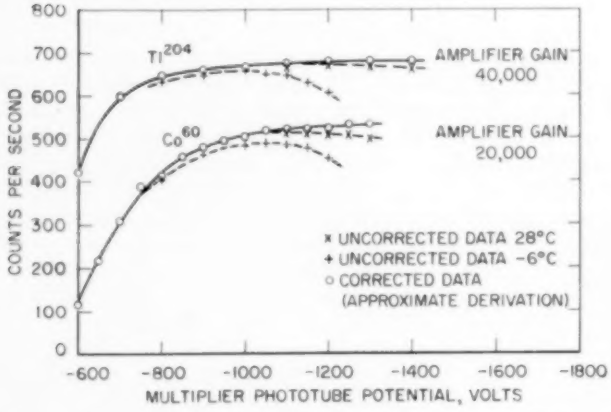

FIGURE 4. *Uncorrected data and corrected values of the absolute disintegration rates for sources of cobalt-60 and thallium-204.*

It is probably the case that for low-energy beta particles the photon efficiency of crystal anthracene will be lower than 1 photon per 65 ev [5]. If one assumes an average efficiency of 95 percent for the 4π sandwich-crystal method in the case of Co^{60} it is possible to make some estimate of this value for anthracene for low-energy beta particles. Ninety-five-percent coincidence efficiency corresponds to 97.5-percent single-channel efficiency or a 2.5-percent loss of low-energy beta particles. As roughly 2.5 percent of the Co^{60} beta spectrum is below 5 kev in energy this would correspond to a minimum energy cutoff of 5 kev. Again assuming a 5-percent photoelectric efficiency for the phototube photocathode one obtains a photon efficiency for anthracene of the order of 1 photon per 120 ev in the range 0 to 5 kev. This is not an unreasonable figure, for in this range the specific ionization of electrons is a very rapidly rising function in the direction of decreasing energy.

TABLE 1. *Summary of data for various nuclides*

Nuclide	$4\pi\beta$-gas counting	4π-crystal counting	Ratio $\frac{\text{Crystal}}{\text{Gas}}$
	dps	*dps*	
P^{32}	274 ±1	275 ±3	1.00
	627 ±3	628 ±3	1.00
Co^{60}	619 ±3	607 ±5	0.98
	609 ±2	597 ±10	.98
	573 ±2	530 ±5	.93
$Sr^{90}-Y^{90}$	880 ±4	884 ±4	1.00
	840 ±4	832 ±8	0.99
I^{131}	741 ±3	736 ±5	.99
	277 ±1	273 ±2	.99
Tl^{204}	371 ±1	373 ±3	1.00
	681 ±2	680 ±3	1.00

4. Conclusions

The 4π-crystal-scintillation technique is suitable for the standardization of medium and high-energy beta emitters and in this energy range gives results that agree quite well with results obtained with the $4\pi\beta$-gas-proportional counter. One possible advantage of the former method is the higher density of anthracene relative to the counter gas and the consequently higher efficiency for the detection of X-rays from electron-capturing nuclides. This may prove useful in the standardization of electron-capturing nuclides.

The authors gratefully acknowledge the assistance of Miss L. M. Cavallo, who prepared many of these sources and who performed the $4\pi\beta$-gas-proportional counting measurements.

WASHINGTON, July 3, 1956.

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Journal of Research of the National Bureau of Standards Vol 57, No. 5, November 1956 Research Paper 2717

Efficiency of 4π-Crystal-Scintillation Counting: 2. Dead-Time and Coincidence Corrections

W. B. Mann and H. H. Seliger

The dead-time and coincidence corrections to be applied in 4π-crystal-scintillation counting have been both approximately and rigorously derived. It has been shown that under special conditions the non-randomness due to "true" counts appearing in both channels requires additional correction terms.

1. Introduction

Two methods of using the 4π-crystal-sandwich-scintillation counter, initiated by one of us [1] [1] for the standardization of radioactive sources, have been described, namely, the "addition" and "coincidence" methods. The former may be used where the counts due to noise in the multiplier phototubes are low compared with the counts due to the radioactive source, whereas the latter method is applied where the "true" counts due to the source are low compared with the noise counts and it is desired to reduce the effect of these random noise counts.

In the addition method the dead-time correction is completely orthodox, it being necessary only to allow for the true counts lost due to the total dead time arising from a given number of counts in the one channel, each count having associated with it a gate or dead time τ_g.

2. Approximate Derivation

In the coincidence method, however, a non-random element is introduced that makes the calculation of the dead-time and coincidence losses somewhat more difficult. But even so, provided that the counting-rate losses due to both noise and source, or true, counts are low, an approximate approach to the problem may be used.

Let us therefore assume, in the first place, that in the coincidence method, the counting rates due to noise in the two channels (N_n' and N_n'' per second) and that due to the radioactive source (N_t per second) are such that there is no appreciable interaction. If, further, N_{tc} is the number of true counts per second traversing each channel and being recorded as coincidences, while $N(\mathrm{I})$ and $N(\mathrm{II})$ are the total number of counts observed in each channel respectively, then

$$N(\mathrm{I})=N_n'+N_{tc} \tag{1}$$

and

$$N(\mathrm{II})=N_n''+N_{tc}. \tag{2}$$

If N_{tc} were extremely small, then the dead times in each channel for a gate dead time of τ_g would be $N_n'\tau_g$ and $N_n''\tau_g$, and the probabilities of detecting a true count in channels I and II would be, respectively,

$$P_{\mathrm{I}}=(1-N_n'\tau_g) \tag{3}$$

and

$$P_{\mathrm{II}}=(1-N_n''\tau_g). \tag{4}$$

As these noise counts are completely random, the probability of a true count being observed in both channel I *and* channel II, that is to say as a recorded coincidence, is given by the multiplication together of the probabilities given in eq (3) and (4). That is,

$$\begin{aligned} P_c^n&=(1-N_n'\tau_g)(1-N_n''\tau_g)\\ &=1-(N_n'+N_n'')\tau_g+N_n'N_n''\tau_g^2, \end{aligned} \tag{5}$$

the last term on the right-hand side representing the overcorrection on account of overlapping of noise counts in the two channels.

If, on the other hand, there were negligible noise counts but a number of true counts N_{tc} per second, then the dead-time correction would be represented by a probability P_c^t, where

$$P_c^t=1-N_{tc}\tau_g. \tag{6}$$

When the quantities $N_n'\tau_g$ and $N_n''\tau_g$ are small, as in the conditions of the present experiments, it can be assumed that there is no appreciable interaction between noise and true counts, and that the total probability of observing a true count as a coincidence is given by the product of the probabilities of eq (3), (4), and (6),

$$P_c\sim 1-(N_n'+N_n''+N_{tc})\tau_g+N_n'N_n''\tau_g^2, \tag{7}$$

neglecting $N_{tc}N_n'\tau_g^2$ and $N_{tc}N_n''\tau_g^2$, which are usually an order of magnitude less than $N_n'N_n''\tau_g^2$. $N_n'N_n''\tau_g^2$ itself involves a correction of at most 0.5 percent, even at high noise counting rates.

[1] Figures in brackets indicate the literature references at the end of this paper.

Substituting for N_n' and N_n'' from eq (1) and (2), we have

$$P_c \sim 1-(N(\mathrm{I})+N(\mathrm{II})-N_{tc})\tau_g+N_n'N_n''\tau_g^2 \quad (8)$$

and

$$N_{tc}=N_t\{1-(N(\mathrm{I})+N(\mathrm{II})-N_{tc})\tau_g+N_n'N_n''\tau_g^2\}. \quad (9)$$

Equation (8) neglects an overcorrection of the form $(\tau_c/2)(N_n'+N_n'')$, where τ_c is the resolving time of the coincidence analyzer. However, as $\tau_c \ll \tau_g$ and we have already assumed that $N_n'\tau_g$ and $N_n''\tau_g$ are small, the approximation should still be a good one.

The coincidence count will also include, however, accidental coincidences arising from the random noise counts. These accidental coincidences are equal to $2\tau_c N_n' N_n''$ per second, where τ_c is the resolving time of the coincidence analyzer. Therefore, the observed coincidence-counting rate of N_c per second is related to the true-count coincidence rate by the expression,

$$N_c=N_{tc}+2\tau_c N_n' N_n''. \quad (10)$$

From eq (9)

$$N_t=\frac{N_{tc}}{1-(N(\mathrm{I})+N(\mathrm{II})-N_{tc})\tau_g+N_n'N_n''\tau_g^2}. \quad (11)$$

N_{tc} is not directly observable because the observed coincidence-counting rate, N_c, is the sum of N_{tc} and the accidental rate due to noise. Substituting for N_{tc} from eq (10) into eq (11) we have that

$$N_t=\frac{N_c-2\tau_c N_n' N_n''}{1-\{N(\mathrm{I})+N(\mathrm{II})-(N_c-2\tau_c N_n' N_n''\}\tau_g+N_n'N_n''\tau_g^2}. \quad (12)$$

In a blank experiment (*without* a source in the crystal-sandwich scintillator)

$$(N_c)_b=(N_{tc})_b+2\tau_c(N_n')_b(N_n'')_b. \quad (13)$$

In this expression, $(N_{tc})_b$ will arise from cosmic radiation, possible contamination of the crystal faces, and fluorescence in the glass of the multiplier phototubes due to potassium–40. However, in the individual channels, $(N_{tc})_b \ll N(\mathrm{I})_b$, $N(\mathrm{II})_b$, so that within a few tenths of a percent,

$$N_n'=(N_n')_b=(N(\mathrm{I}))_b-(N_c)_b+2\tau_c(N(\mathrm{I})_b N(\mathrm{II})_b) \quad (14)$$

and

$$N_n''=(N_n'')_b=(N(\mathrm{II}))_b-(N_c)_b+2\tau_c(N(\mathrm{I})_b N(\mathrm{II})_b). \quad (15)$$

Substitution of these values for N_n' and N_n'' in eq (12), and subtraction of $(N_{tc})_b$, as given by eq (13), from N_t, as calculated from eq (12), yields the true disintegration rate of the source to within a few tenths of a percent.

This approximate derivation of the dead-time losses is, however, based on certain premises that may not always be strictly applicable in practice. In the first place, if the noise counts are negligible, the dead-time losses would be given exactly by eq (6); but in practice the noise counts can never be neglected. If, on the other hand, the true counts are very low, the correction for dead-time losses would be given by eq (5); but then the method would be impractical, as long periods of counting would be required to give good statistics. The third possibility that has been assumed in the derivation, namely, that of reasonable numbers of noise and true counts but limited interaction between the two due to low counting rates, reflects the conditions of the present experiments.

It was felt that several of the assumptions made in the above approximate derivation are in a measure intuitive, and that the experimental results obtained would be placed on a more firm footing if a rigorous derivation were developed to establish the relative orders of magnitude of the various interactions. In addition, once one enters into the realm of very high counting rates with large counting-rate losses, there will be an interaction between noise and true counts, and it will be impossible to multiply the probabilities P_c^n and P_c^t to give the combined probability of observing a true count coincidence, P_c, of eq (7). Moreover, because a great many true counts in channel I are linked to true counts in channel II, it will not be permissible to derive the probability by multiplying together two individual probabilities for the separate channels as was done to derive P_c^n (eq (5)) from P_I and P_II (eq (3) and (4)), although the individual probabilities P_I and P_II do refer to processes in channel I and in channel II that are completely random.

3. Rigorous Derivation

In determining the dead-time loss and accidental-coincidence corrections that must be applied in the method of 4π-crystal-scintillation counting, it is necessary to apply a rigorous analysis of the events occurring in the two multiplier-phototube channels that lead to the coincidence analyzer. This is necessitated by the fact that a number of the events occurring in one channel are linked to those occurring in the other, and it may be therefore no longer valid to apply the kind of considerations that would be possible if all the events were completely randomly distributed.

Let us assume that N_t counts per second, the so-called true counts, arise in the composite crystal scintillator due to the radioactive source and the background. Then, of these counts, a number N_{tc} per second will be recorded by the coincidence analyzer. Further, let N_n' and N_n'' be the number of noise counts per second passing along channel I and channel II, respectively, and let N_c be the total number of coincidences recorded per second by the coincidence analyzer.

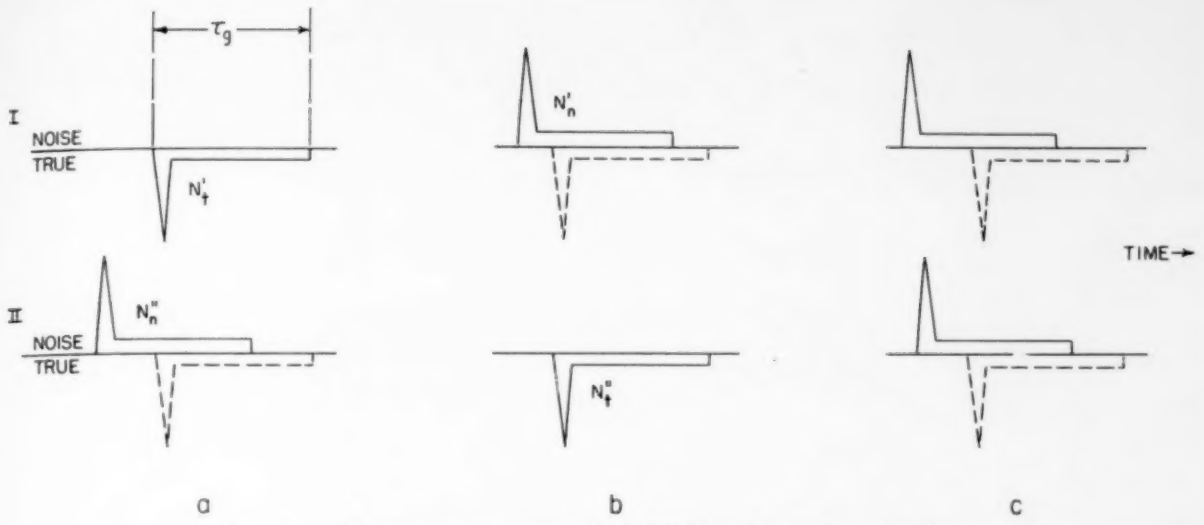

FIGURE 1. *Diagrammatic representation of different kinds of counting losses.*

It is, however, also necessary to consider, in each channel, what may be conveniently designated as "divorced true counts" which we will assume to be respectively N'_t and N''_t per second. These arise in the following manner. A true count, if it can pass through both channels unimpeded, arrives at the coincidence analyzer as a "perfect" coincidence with no delay whatsoever and, as an "N_{tc}" event, will be recorded. If, however, the pulse due to a true count happens to coincide with the gate of a preceding noise pulse in one channel, the true count in that channel will be lost and the pulse in the other channel due to the true count will proceed to the coincidence analyzer linked to the noise pulse that has "consumed" its partner. A divorced true count is thus always linked temporally to a noise count in the other channel, and such divorced true counts are therefore no longer randomly distributed with respect to the other counts.

The diagrams in figure 1 will help to illustrate the relationships of the various kinds of counts one to another. In these diagrams the pulses of noise counts are shown above the time axis of each channel, whereas the pulses due to true counts are shown below the axes, although in reality they are experimentally indistinguishable. Figure 1(a) illustrates the circumstances in which a divorced true count (N'_t per second) can arise in channel I, figure 1(b) the circumstances giving rise to a divorced true count (N''_t per second) in channel II, and figure 1(c) those in which the true counts can be lost in both channels I and II. In all cases the length of the pulse is chosen to signify the time τ_g in which the electronic gates in either channels I or II will be closed to the transmission of further pulses.

What then are the numbers of divorced true counts, N'_t and N''_t per second, in channels I and II, respectively?

These correspond to the numbers of overlaps between true-count pulses and *preceding* noise-count pulses in the *other* channel, for times greater than τ_e, the resolving time of the coincidence analyzer.

If a pair of true counts is impressed on the circuit within a time interval less than τ_e after a single noise pulse or a coincident pair of noise pulses, the coincidence analyzer will accept these as a true coincidence. The effective noise-pulse dead time for the creation of divorced counts in the other circuit is thus $\tau_g - \tau_e$.

In channel II, the noise counts N''_n per second give rise, therefore, to an effective total dead time of $N''_n(\tau_g - \tau_e)$ per second for the creation of divorced true counts (N'_t per second) in channel I. During this time N_t true counts will be submitted to channels I and II. Therefore, if there were no interaction between channels I and II,

$$\text{divorced true counts in channel I} = (\tau_g - \tau_e)N''_n N_t \text{ per second.} \quad (16)$$

Similarly,

$$\text{divorced true counts in channel II} = (\tau_g - \tau_e)N'_n N_t \text{ per second.} \quad (17)$$

These expressions as they stand give, however, somewhat higher values than N'_t and N''_t, because they also include cases of overlap, as in figure 1(c), in which both true-count pulses are rejected by two *preceding* noise counts.

The numbers of overlaps between a pair of true-counts and a preceding noise count in channels I and II are, respectively, $(\tau_g - \tau_e)N'_n N_t$ and $(\tau_g - \tau_e)N''_n N_t$ per second.

The time in which a second noise count in channel II (fig. 2) must occur to eliminate N''_t varies therefore from 0 (fig. 2(a)) to $(\tau_g - \tau_e)$ (fig. 2(b)), giving an average time per overlap of $\frac{1}{2}(\tau_g - \tau_e)$.

Thus, considering figure 2, the number of such divorced true counts passing along channel II is $(\tau_g - \tau_e)N'_n N_t$ per second (eq 17), and the average dead time for each of these events for the acceptance of a *preceding* noise count in channel II which will eliminate the "N''_t" pulse is $\frac{1}{2}(\tau_g - \tau_e)$. This corre-

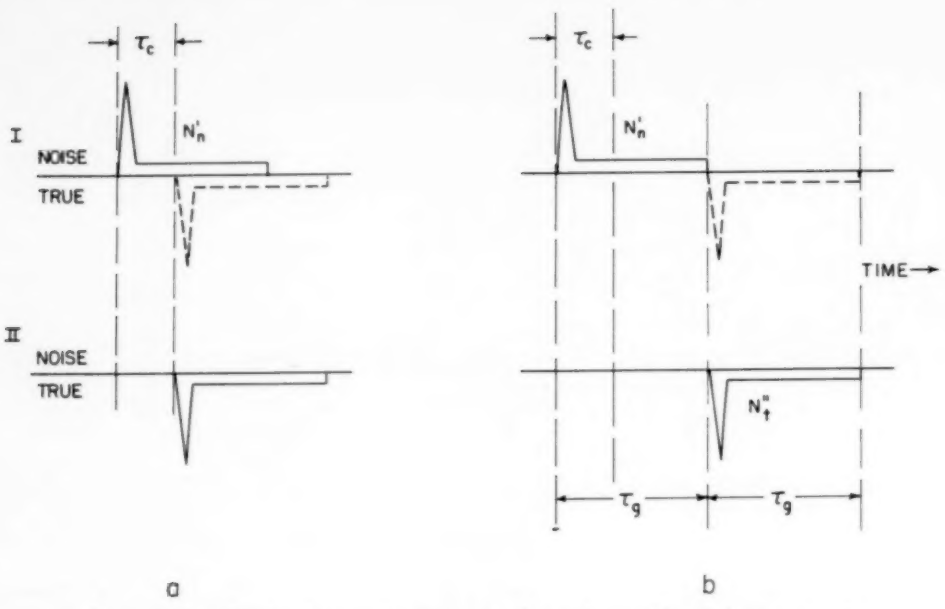

FIGURE 2. *Overlap between true-count pulse and preceding noise-count pulse.*

sponds for such events to a dead time per second of $\frac{1}{2}(\tau_g-\tau_e)^2N'_nN_t$. With N''_n noise counts per second in channel II, the number of triple events causing the loss of both of a pair of true counts in channel I and channel II is $\frac{1}{2}(\tau_g-\tau_e)^2N'_nN''_nN_t$ per second. It is seen that the same result will be obtained by proceeding via "N'_t" events in channel I (eq 16) and the N'_n counts in channel I.

We thus derive the following corrected values for the divorced true counts in channels I and II:

$$N'_t=(\tau_g-\tau_e)N_tN''_n\{1-\tfrac{1}{2}(\tau_g-\tau_e)N'_n\} \text{ per second,} \tag{18}$$

and

$$N''_t=(\tau_g-\tau_e)N_tN'_n\{1-\tfrac{1}{2}(\tau_g-\tau_e)N''_n\} \text{ per second.} \tag{19}$$

It is now possible to proceed to the computation of the true counts lost as a result of the events that are summarized in figure 1. In the first place, every "N'_t" and "N''_t" event corresponds to a lost true-count pair. In addition, true counts can be lost in the dead times initiated in each channel by the divorced true counts, N'_t and N''_t per second. The dead times involved in "N'_t" or "N''_t" events vary from τ_g to $\tau_g-\tau_e$, as illustrated for the "N''_t" events in figure 3.

The dead time is only that which *succeeds* an "N'_t" or "N''_t" pulse. In no case does the *preceding* noise count contribute to the dead time, for, from the very nature of the divorced true counts, an earlier true-count pair following a noise count merely creates an earlier divorced true count.

It is a curious fact that, as shown in figure 3(c), an intermediate time interval, varying from 0 to τ_e, becomes available for acceptance of true counts immediately following the noise-count dead time in channel I. This is effected when the channel-I pulse of the true-count pair registers as a coincidence with the "N''_t" pulse at the coincidence analyzer. The fraction of the "N'_t" or "N''_t" events in which this window becomes available is τ_e/τ_g, and its width is on the average equal to $\frac{1}{2}\tau_e$. Thus, the true counts that can be accepted in this window for "N'_t" and "N''_t" events are, respectively, $\frac{1}{2}\tau_e^2/\tau_gN'_tN_t$ and $\frac{1}{2}\tau_e^2/\tau_gN''_tN_t$ per second.[2]

It therefore follows that the total true counts lost in channels I and II due to the dead times succeeding "N'_t" and "N''_t" events are respectively $N'_tN_t(\tau_g-\frac{1}{2}\tau_e^2/\tau_g)$ and $N''_tN_t(\tau_g-\frac{1}{2}\tau_e^2/\tau_g)$ per second.

In addition to losses of true counts N'_t, N''_t, $N'_tN_t(\tau_g-\frac{1}{2}\tau_e^2/\tau_g)$, and $N''_tN_t(\tau_g-\frac{1}{2}\tau_e^2/\tau_g)$, true counts can also be lost in the triple events illustrated by figure 1(c). These number $\frac{1}{2}(\tau_g-\tau_e)^2N'_nN''_nN_t$ per second. These triple events in which a pair of true counts is completely eliminated by *two* noise counts, one in each channel, are illustrated in greater detail in figure 4.

As can be seen from inspection of figure 4, the dead time due to the second noise count varies from τ_g (fig. 4(a)) to the average of τ_e and τ_g, or $\frac{1}{2}(\tau_g+\tau_e)$ (fig. 4(b)). This is equivalent to a total average dead time of $\frac{1}{4}(3\tau_g+\tau_e)$. The true counts lost in triple events such as that of figure 1(c) are therefore $\frac{1}{8}(\tau_g-\tau_e)^2\ (3\tau_g+\tau_e)N'_nN''_nN_t^2$ per second. Therefore, the total true counts lost per second as a result of

[2] In the actual experiment the resolving time of the amplifier was greater than τ_e, so that coincidences such as are indicated in figure 3(c) do not occur. Figure 3(c) is valid only for very sharp pulses whose duration is much less than τ_e.

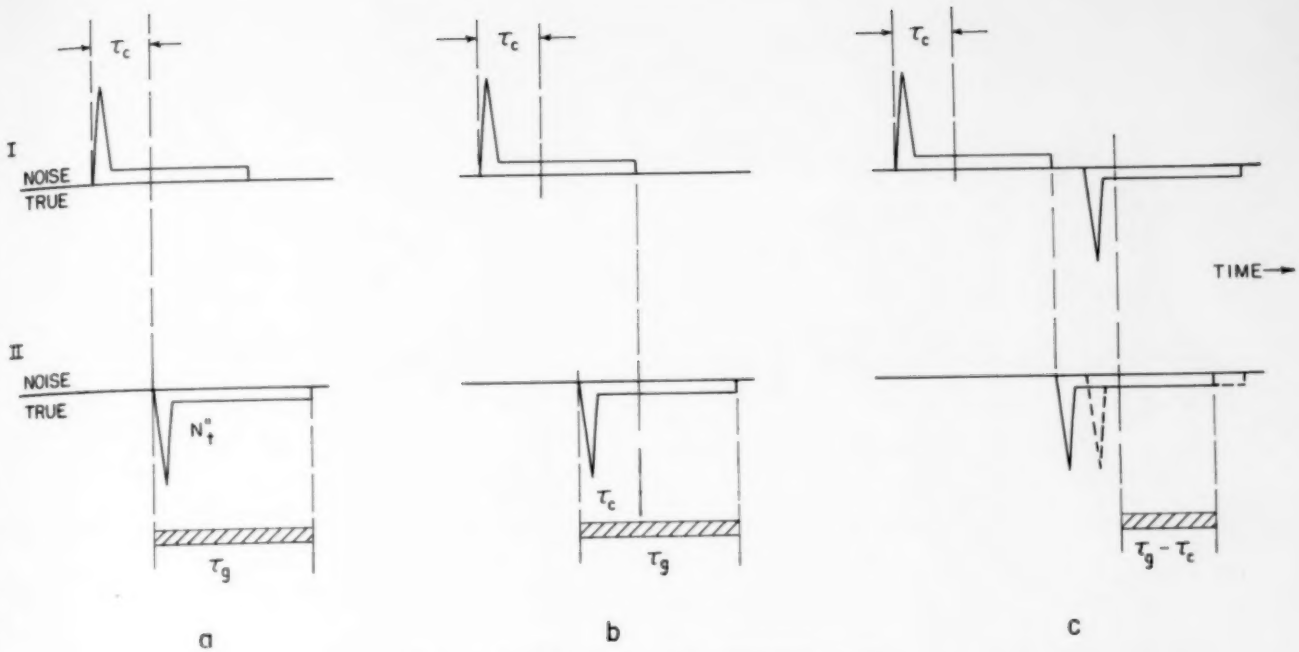

FIGURE 3. *Dead times of divorced true counts.*

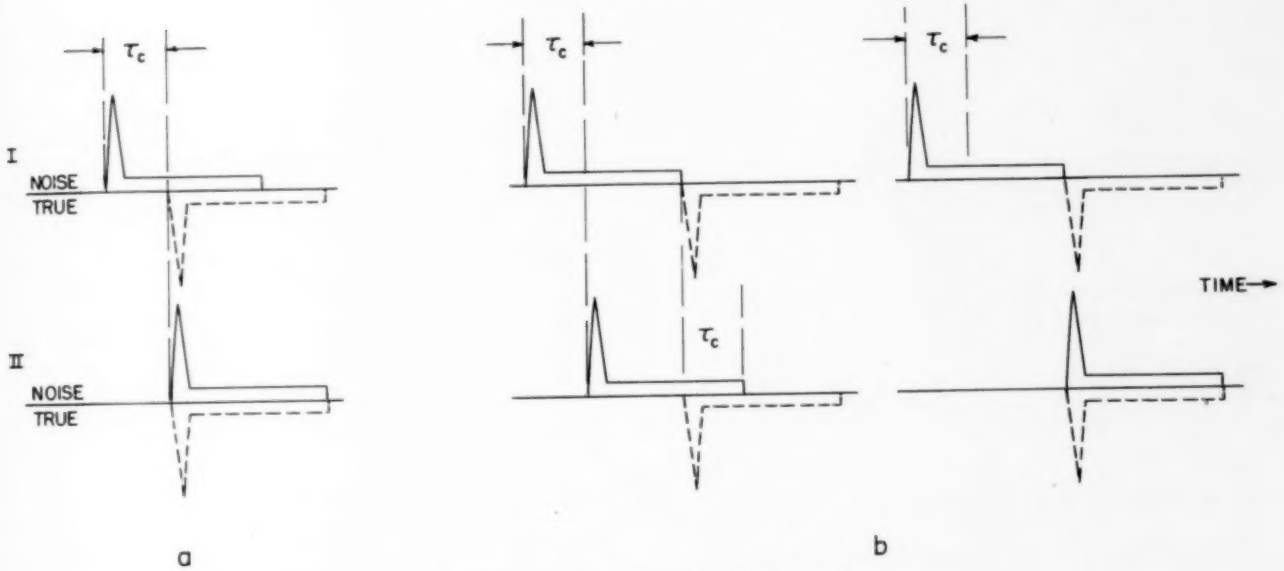

FIGURE 4. *Dead times of "triple events."*

losses in both channels I and II due to overlapping true-count and noise-count pulses are

$$\Sigma N_{tl}=N'_t+N''_t+N_t\left(\tau_g-\tfrac{1}{2}\frac{\tau_c^2}{\tau_g}\right)(N'_t+N''_t)$$
$$+\tfrac{1}{8}(\tau_g-\tau_c)^2(3\tau_g+\tau_c)N'_nN''_nN_t^2.$$

$$\therefore \Sigma N_{tl}=(N'_t+N''_t)\left\{1+N_t\left(\tau_g-\tfrac{1}{2}\frac{\tau_c^2}{\tau_g}\right)\right\}$$
$$+\tfrac{1}{8}(\tau_g-\tau_c)^2(3\tau_g+\tau_c)N'_nN''_nN_t^2. \quad (20)$$

$$\sum\frac{N_{tl}}{N_t}=(\tau_g-\tau_c)\left\{N'_n(1-\tfrac{1}{2}(\tau_g-\tau_c)N''_n)\right.$$
$$\left.+N''_n\{(1-\tfrac{1}{2}(\tau_g-\tau_c)N'_n)\right\}\left\{1+N_t\left(\tau_g-\tfrac{1}{2}\frac{\tau_c^2}{\tau_g}\right)\right\}$$
$$+\tfrac{1}{8}(\tau_g-\tau_c)^2(3\tau_g+\tau_c)N'_nN''_nN_t. \quad (21)$$

N_{tc} is the number of true-count coincidences per second that are registered by the coincidence analyzer and therefore pass through channels I and II unimpeded. Each of these counts imposes a gate

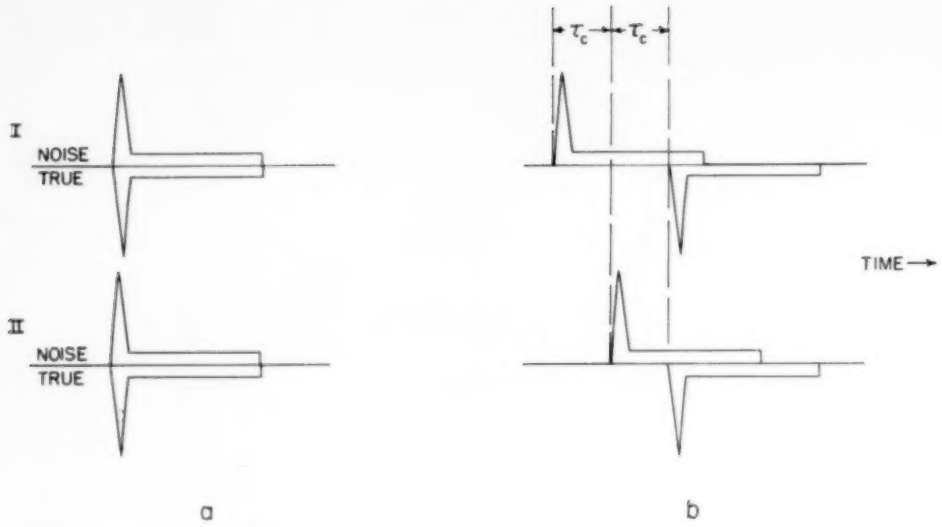

FIGURE 5. *Coincidences between accidental noise coincidences and true counts.*

dead time of τ_g, and gives rise therefore to $N_{te}\tau_g N_t$ losses per second. Therefore, the total of all true counts lost from any cause whatsoever is equal to $N_{te}N_t\tau_g+\sum N_{tl}$.

The true counts per second, N_t, are therefore given by

$$N_t=N_{te}+N_{te}N_t\tau_g+\sum N_{tl}$$

$$N_t=\frac{N_{te}}{1-N_{te}\tau_g-\dfrac{\sum N_{tl}}{N_t}}, \tag{22}$$

N_{te} being the number of true-count pairs that pass through channels I and II unimpeded. N_c is, however, the *total* number of coincidence counts recorded per second. Thus the difference between N_{te} and N_c will be essentially due to accidental noise coincidences that are of the order $2\tau_c(N_n'-N_t'')(N_n''-N_t')$ per second. Thus,

$$N_{te}\sim N_c-2\tau_c(N_n'-N_t'')(N_n''-N_t'). \tag{23}$$

The reason for subtracting N_t'' from N_n' and N_t' from N_n'' in the accidental noise-coincidence term is that any noise count associated with an "N_t''" or "N_t'''" event can *never* contribute to an accidental coincidence. This accidental coincidence correction in eq (23) is, however, an overcorrection in that true-count pulses can arrive during the time the accidental noise coincidences are being accepted by the coincidence analyzer. The time during which such true counts can be accepted as coincidences is seen by inspection of figure 5 to vary from 0 to $2\tau_c$, giving an average of τ_c.

The number of coincidences between accidental noise coincidences and true-count pulses is thus equal to $2\tau_c(N_n'-N_t'')(N_n''-N_t')\tau_c N_t$ per second. In other words, the accidental noise-coincidence term must be multiplied by a factor equal to $(1-\tau_c N_t)$ to allow for true counts coinciding with accidental noise coincidences. Thus, as a closer approximation,

$$N_{te}\sim N_c-2\tau_c(N_n'-N_t'')(N_n''-N_t')(1-\tau_c N_t). \tag{24}$$

There is, however, still one other possible source of coincidence counts that cannot be overlooked. If we consider figure 1(a) in detail it will be seen, as shown in figure 6, that a window also exists for a very small number of coincidences of "N_t'" and "N_t''" events with noise counts, just as a window was previously noted to exist (fig. 3(c)) for the acceptance of true counts.

The numbers of "N_t'" and "N_t''" events that can be available for such coincidences are equal respectively to $[\tau_c/(\tau_g-\tau_c)]N_t'$ and $[\tau_c/(\tau_g-\tau_c)]N_t''$ for $2\tau_c<\tau_g$ or N_t' and N_t'' for $2\tau_c>\tau_g$. The average time that the window is open for the acceptance of a noise count in coincidence is $\frac{1}{2}\tau_c$, provided that $2\tau_c<\tau_g$, or $\frac{1}{2}(3\tau_c-\tau_g)$ if $2\tau_c>\tau_g$. The numbers of such coincidences for "N_t'" and "N_t''" events, respectively, are therefore $\frac{1}{2}[\tau_c^2/(\tau_g-\tau_c)]N_t'N_n''$ and $\frac{1}{2}[\tau_c^2/(\tau_g-\tau_c)]N_t''N_n'$ per second as, in general, $2\tau_c<\tau_g$. As these coincidences can also be in coincidence with true-count pulses, they too represent an overcorrection and must be multiplied by the same correction factor $(1-\tau_c N_t)$ used to correct the accidental noise-coincidence counts of eq (24).[3]

We can therefore now write exactly that

$$N_{te}=N_c-\left\{2\tau_c(N_n'-N_t'')(N_n''-N_t')+\frac{1}{2}\frac{\tau_c^2}{\tau_g-\tau_c}(N_t'N_n''+N_t''N_n')\right\}(1-\tau_c N_t). \tag{25}$$

[3] Here again, however, because the resolving time of the amplifier was greater than τ_c, these coincidences could not, in practice, occur.

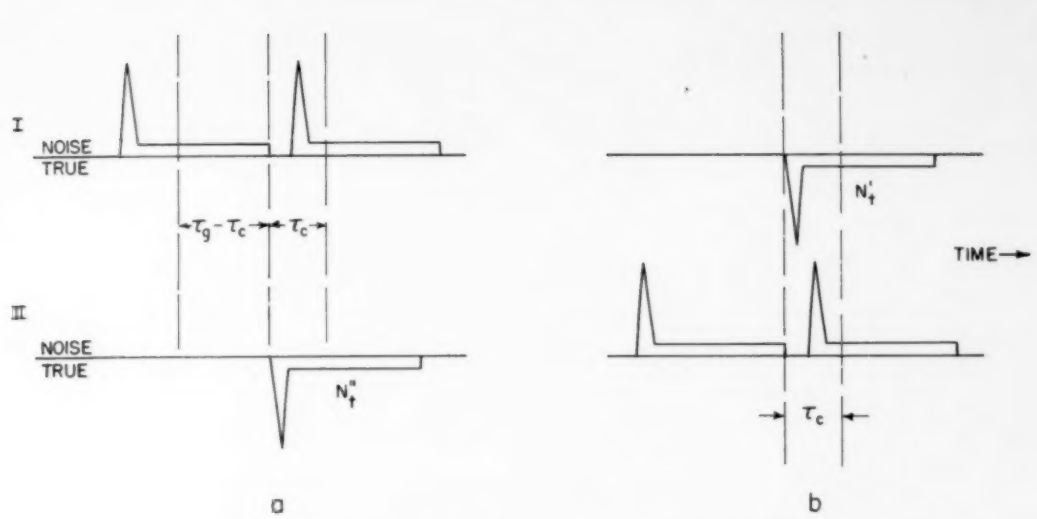

FIGURE 6. *"Window" for coincidences between noise counts and divorced true counts.*

The true counts per second, N_t, are now given by

$$N_t = \frac{N_{te}}{1 - N_{te}\tau_g - \dfrac{\Sigma N_{t1}}{N_t}}, \tag{22}$$

where N_{te} is given by eq (25) and $\dfrac{\Sigma N_{t1}}{N_t}$ by eq (21).

The expression for N_t can therefore be written down exactly for those who desire to experience such an exercise or whose experimental conditions demand it. In the interest of economy of space, however, let us at this point make an approximation that is justified by the physical quantities involved under the experimental conditions [1], namely, to neglect all terms involving higher powers than τ_g^2 ($\tau_g \simeq 10$ μsec) and τ_c (about $\frac{1}{2}$ μsec) in eq (21). Then, by substitution from eq (21) into eq (22),

$$N_t = \frac{N_{te}}{1 - N_{te}\tau_g - (\tau_g - \tau_c)\{N_n' + N_n'' - (\tau_g - \tau_c)N_n'N_n''\}\{1 + N_{te}\tau_g\}}. \tag{26}$$

It is seen from eq (18) and (19) that substitution for N_t' and N_t'' in eq (25) for N_{te} will introduce terms involving $\tau_c(\tau_g - \tau_c)$ and $\tau_c(\tau_g - \tau_c)^2$, which can be neglected by comparison with $2\tau_c N_n' N_n''$. Equation (26) can therefore be further simplified to give

$$N_t = \frac{N_c - 2\tau_c N_n' N_n''}{1 - \tau_g(N_c - 2\tau_c N_n' N_n'') - (\tau_g - \tau_c)\{N_n' + N_n'' - (\tau_g - \tau_c)N_n' N_n''\}}. \tag{27}$$

As a further simplification, because

$$N_n' + N_n'' \simeq N(\mathrm{I}) + N(\mathrm{II}) - 2(N_c - 2\tau_c N_n' N_n''),$$

if all products of τ's containing τ_c are neglected,

$$N_t \simeq \frac{N_c - 2\tau_c N_n' N_n''}{1 - \tau_g N_c - (\tau_g - \tau_c)\{N(\mathrm{I}) + N(\mathrm{II}) - 2N_c\} + \tau_g^2 N_n' N_n''}. \tag{28}$$

In order to determine N_n', and N_n'' a blank experiment can be carried out with the crystal removed and with an opaque obstruction between the two multiplier photo tubes, to prevent phosphorescence in one photocathode causing a simultaneous signal in both. Then N_t is zero and the noise counts will be registered directly by the scalers in each of channels I and II. By inserting the crystal and carrying out experiments with and without a source, the values for N_t, for both the source plus background and background, respectively, can then be determined in terms of the number of coincidences and the known quantities τ_c and τ_g and the noise counts.

It is interesting to note from an inspection of eq (18) and (19) that if $\tau_g = \tau_c$, the numbers of divorced counts N_t' and N_t'' become zero. For such a condition, ΣN_{t1} (from eq (21)) also becomes zero and eq (22) gives without any approximation

$$N_t = \frac{N_c - 2\tau_c N_n' N_n''(1 - \tau_c N_t)}{1 - N_c\tau_c + 2\tau_c N_n' N_n''(1 - \tau_c N_t)}. \tag{29}$$

In considering the case for $\tau_g = \tau_c$, it is also necessary to use the form of eq (25), for N_{te}, where $2\tau_c > \tau_g$, namely,

$$N_{te} = N_c - \Big\{ 2\tau_c\,(N_n' - N_n'')(N_n'' - N_t') + \frac{1}{2}(3\tau_c - \tau_g)(N_t' N_n'' + N_t'' N_n') \Big\}\Big(1 - \tau_c N_t\Big), \tag{30}$$

in which, however, the second term in the first bracket becomes zero when $\tau_g = \tau_c$ by virtue of both N_t' and N_t'' becoming zero. However, this is never experimentally feasible, for (a) if τ_c is *increased* to equal τ_g, the accidental rate will be too high, and (b) if τ_g is *decreased* to equal τ_c, the non-randomness tends to disappear and the rigorous derivation becomes unnecessary.

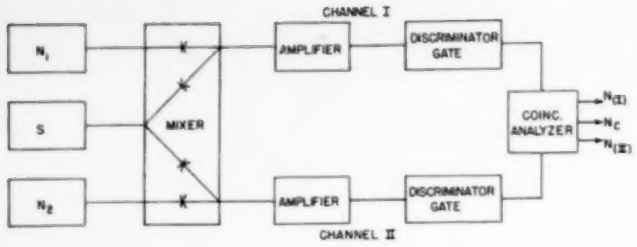

FIGURE 7. *Experimental arrangement for testing validity of rigorous and approximate derivations of the dead-time and coincidence losses.*

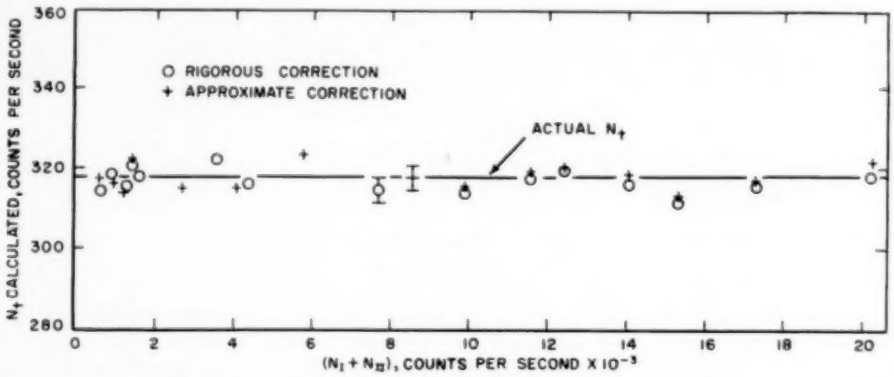

FIGURE 8. *"True" counts after correction by both the rigorous and approximate methods.*
Limits of error are indicated on only two points.

4. Comparison of Rigorous and Approximate Derivations

It is of interest to compare experimentally the validity of the corrections required by the approximate derivation, eq (12), and the rigorous derivation, eq (28). The arrangement shown schematically in figure 7 was designed to test directly both derivations over the range of phototube noise encountered in the experiments.

A sandwich source of Tl^{204}, covered by a hemispherical aluminum reflector, was counted by phototube S. These counts, *and the noise counts* from phototube S, were fed into channels I and II *simultaneously*, thus providing a source of "true" pulses. Separate phototubes, N_1 and N_2, were then used as independent sources of random noise, and pulses from N_1 and N_2 were fed at the same time into channels I and II, respectively. The counting rate from S was maintained constant.

The voltages across the dynodes of N_1 and N_2 were varied stepwise so as to increase the random noise from 0 to 11,000 counts per second, covering the most extreme noise measurements encountered in the actual experiments. The noise pulses from N_1 and the "true" pulses from S passed through channel I, and similarly the noise pulses from N_2 and the same "true" pulses from S passed through channel II. $N(\mathrm{I})$, $N(\mathrm{II})$, and N_c, were then analogous to the values obtained in the actual experiment, except in this case N_t was known exactly (by counting S alone) and N_n' and N_n'' were known exactly (by counting N_1 and N_2 separately). In figure 8 are shown the values of N_t (approximate) and N_t (rigorous) obtained by means of eq (12) and (28), respectively, as functions of $N(\mathrm{I})+N(\mathrm{II})$, the sum of the single-channel counting rates. As can be seen, both methods of correction are applicable over the range of phototube noise encountered in the experiment.

The foregoing derivations will be applicable to any system where coincidence between two counters is used to select particular events from a high-background environment.

We gratefully acknowledge the assistance of Carter C. Smith in carrying out the experimental comparison of the rigorous and approximate derivations.

5. Reference

[1] C. C. Smith, H. H. Seliger, and J. Steyn, J. Research NBS **57**, 251 (1956) RP2716.

WASHINGTON, July 3, 1956.

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Entropy Changes in Rarefaction Waves[1]

Robert F. Dressler

Frictional flow of a polytropic gas is investigated for centered rarefaction waves. Energy balance is maintained by assuming that mechanical energy loss due to frictional force reappears as heat. Entropy behavior at the wave front is discussed. The nonhomogeneous linear system of partial differential equations with variable coefficients for first-order frictional effects is derived. Utilizing the geometrical similarity of the Mach lines, the functional form of these quantities is ascertained, which permits explicit solutions of the boundary value problem. The expansion procedure is singular at the wave front, and results are not applicable there. First-order effects are expressed as polynomials plus terms singular at the wave front. Results are compared with expressions obtained when heat generation due to frictional force is neglected.

1. Introduction

Equations for one-dimensional, unsteady flow of a compressible gas permit explicit solution for centered rarefaction waves, such as occur in shock tubes or pneumatic control circuits. For some applications, however, it becomes important to include also the complications resulting from frictional dissipation. This matter is discussed here in two ways, first with the simplifying assumption that the frictional effects consist merely in a retarding force, and second, by the more complete consideration that includes the resulting heat generation and accompanying change in entropy. This means that the mechanical energy destroyed by the retardation force reappears as heat energy, resulting in additional expansion. The simpler flow description without heat generation will be referred to as model A, and the second, more accurate, one as model B. The equations for steady frictional flow in model B have been studied and yield energy theorems and an extended form of Bernoulli's theorem [1],[2] but knowledge of the unsteady flows appears to be restricted mainly to a few isolated numerical calculations.

A centered rarefaction wave is perhaps the most basic pattern in unsteady flow; the present paper derives solutions for first-order dissipative effects in this unsteady wave using the full system of equations of model B, and compares results with corresponding ones for model A (which were previously published by the author [2] as a supplementary result in a study of shock tubes with varying cross section).

A recent paper by Ludford and Martin [3] has discussed anisentropic effects in centered simple waves. They consider flows where the specific entropy varies from particle to particle, but remains constant for a given particle. In frictional flow, however, the specific entropy for a given particle must vary with time, and a centered rarefaction wave cannot remain a simple wave.

Consider a polytropic gas at rest for time $\bar{t}<0$ in a duct of uniform cross section, filling the duct to the left of a diaphragm where $\bar{x}<0$, and with a vacuum initially at $\bar{x}>0$. Heat exchange between the gas and the duct, heat conduction and radiation, and viscous effects are disregarded. When the diaphragm is removed at $\bar{t}=0$, a centered rarefaction wave begins. Dimensional quantities are denoted by bars in order to define the unbarred dimensionless quantities to be used subsequently. Before flow, the gas state is described by sound speed $\bar{c}_0$, mass density $\bar{\rho}_0$, and temperature $\bar{T}_0$; after flow we have velocity $\bar{u}$, local sound speed c, mass density $\bar{\rho}$, and pressure $\bar{p}$.

For model A, assuming no internal heat generation and no entropy changes, the equation of state can be written in forms

$$\bar{p}=\bar{A}^*\bar{\rho}^\gamma, \qquad \bar{c}^2=\gamma\bar{A}^*\bar{\rho}^{\gamma-1}=\gamma\bar{R}\bar{T}, \tag{1}$$

[1] This research was supported by the Office of Naval Research, USN.

[2] Figures in brackets indicate the literature references at the end of this paper.

where γ is the adiabatic exponent, $\overline{R}$ the gas constant, $\overline{T}$ the temperature, and $\overline{A}^*$ is a known constant (consistent with the assumption here of no entropy change). In this simpler model, the friction produces merely a retarding force per unit mass of magnitude $\lambda\overline{u}^2$ where λ is the known friction factor. There is some question concerning the constancy of this coefficient for frictional flows that are highly unsteady; this question is discussed by Schultz-Grunow [4], Jenny [5], and others. Here it is considered to be a constant.

Introducing dimensionless (unbarred) variables defined by $x=\overline{x}/\overline{D}$, $t=(\overline{c}_0/\overline{D})\overline{t}$, $u=\overline{u}/\overline{c}_0$, $c=\overline{c}/\overline{c}_0$, $\rho=\overline{\rho}/\overline{\rho}_0$, $\lambda=\overline{D}\overline{\lambda}$ where $\overline{D}$ is some length associated with duct cross section, the momentum and continuity equations for model A are

$$\left.\begin{aligned} u_t+uu_x+\frac{2}{\gamma-1}\,cc_x&=-\lambda u|u| \\ cu_x+\frac{2}{\gamma-1}\,uc_x+\frac{2}{\gamma-1}\,c_t&=0 \end{aligned}\right\} \tag{2}$$

for the two unknowns $u(x,t)$ and $c(x,t)$; and the energy equation is neglected.

In deriving equations for model B one must take account of the mechanical energy lost through the action of the retarding force $\lambda u|u|$. The action of a body force cannot itself directly change entropy; however, if a term is included in the energy equation for an amount of generated heat equal to the lost mechanical energy, this heat will change the specific entropy. In this way, the energy equation becomes expressible directly in terms of the friction coefficient. The equation of state for model B now includes the entropy dependence,

$$\overline{p}=\overline{A}(\overline{s})\overline{\rho}^{\gamma}, \qquad \overline{A}(\overline{s})=(\gamma-1)e^{(\overline{s}-\overline{s}_0)/\overline{c}_v}, \qquad \overline{c}^2=\gamma\overline{R}\overline{T}(\overline{\rho},\overline{s}) \tag{3}$$

where $\overline{c}_v$ is specific heat at constant volume, $\overline{s}$ is the specific entropy (entropy per unit mass), and $\overline{s}_o$ is a constant with a value depending upon the entropy level adopted. Letting dimensionless specific entropy s be defined by $s=(\overline{T}_0/\overline{c}_0^{\,2})\overline{s}$, the momentum, continuity, and energy equations for model B become, respectively,

$$\left.\begin{aligned} u_t+uu_x+\frac{2}{\gamma-1}\,cc_x&=-\lambda u|u|+c^2s_x \\ cu_x+\frac{2}{\gamma-1}\,uc_x+\frac{2}{\gamma-1}\,c_t&=\lambda\gamma\,\frac{|u|^3}{c} \\ s_t+us_x&=\lambda\,\frac{|u|^3}{c^2} \end{aligned}\right\} \tag{4}$$

for the three unknowns $u(x,t)$, $c(x,t)$, and $s(x,t)$. We set $\overline{s}=s=0$ for the gas when initially at rest, and the equality

$$\overline{A}^*=\overline{A}(0)=(\gamma-1)e^{-\overline{s}_0/\overline{c}_v} \tag{5}$$

determines the constant $\overline{s}_0$.

2. Characteristic Equations

The Mach lines for model B are defined by the system of characteristic equations equivalent to the totally hyperbolic system (4). These consist of three families of real curves with directions given by

$$\left.\begin{aligned} \zeta_1\colon\ \frac{dx}{dt}&=u-c \\ \zeta_2\colon\ \frac{dx}{dt}&=u+c \\ \zeta_3\colon\ \frac{dx}{dt}&=u \end{aligned}\right\} \tag{6}$$

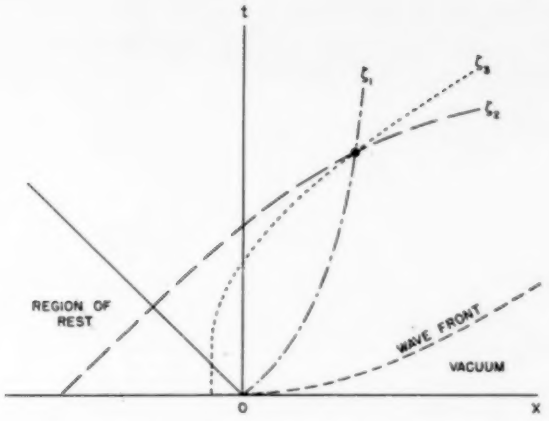

FIGURE 1.

which are respectively the backward sound path, the forward sound path, and the particle trajectory (see fig. 1). The corresponding differential relations along each family are

$$\left.\begin{aligned} \zeta_1\text{:}\quad & d\left(u-\frac{2}{\gamma-1}c\right)+c\,ds=-\lambda u|u|\left(1+[\gamma-1]\frac{u}{c}\right)dt \\ \zeta_2\text{:}\quad & d\left(u+\frac{2}{\gamma-1}c\right)-c\,ds=-\lambda u|u|\left(1-[\gamma-1]\frac{u}{c}\right)dt \\ \zeta_3\text{:}\quad & ds=\lambda\frac{|u|^3}{c^2}\,dt. \end{aligned}\right\} \tag{7}$$

These equations are used in [6] as the basis for a discussion of a numerical computation on a rarefaction wave in a shock tube.

3. Entropy Behavior at the Wave Front

From the above equations and the equation of state (3), the behavior of s at the forward wave front where $\rho=0$ can be inferred, consistent with the assumed model for the flow. For notational simplicity and to permit numerical results, we now specialize the gas to be air with $\gamma=7/5$; however, the same arguments will apply and analogous solutions can be derived for any other admissible value of γ.

If $\overline{s}_0$ is chosen as indicated and the relation $\gamma=1+\overline{R}/\overline{c}_v$, is used, we obtain the dimensionless equation of state in the form $c^2=e^{\gamma(\gamma-1)s}\rho^{\gamma-1}$, which for air is

$$\rho=e^{-7s/5}c^5. \tag{8}$$

On the wave front curve where $\rho=0$, this permits two possibilities: either (a), s remains finite and hence $c=0$ there, or (b), s becomes infinite and c is not known in advance. If s were finite and continuous and c vanished there, then by the third equation in (7), the rate of change of s along the wave front would be everywhere infinite as $u\neq 0$ there. This follows from the fact that the wave-front locus is also a particle trajectory and therefore a ζ_3 Mach line. Therefore s must actually be infinite on the wave-front curve, and c is not determined there by this argument. Because s is entropy per mass, this does not necessarily imply that entropy per volume becomes infinite at the wave front for this model.

4. First-Order Effects

By setting up asymptotic expansions of form

$$\left.\begin{aligned} v(x,t,\lambda)&\sim v^0(x,t)+V^A(x,t)\lambda+\ \ldots\ \text{(model A)} \\ v(x,t,\lambda)&\sim v^0(x,t)+V^B(x,t)\lambda+\ \ldots\ \text{(model B)} \end{aligned}\right\} \tag{8}$$

a perturbation scheme will be defined to give information for small values of λ. The quantities $v(x,t,\lambda)$ represent each unknown u,c,s, and their various first derivatives. The unperturbed centered rarefaction wave is the known simple-wave solution

$$\left.\begin{aligned} u^0&=5(1+x/t)\\ c^0&=\tfrac{1}{6}(5-x/t)\\ s^0&=0 \end{aligned}\right\} \tag{10}$$

which when $\lambda=0$ satisfies system (2) for model A and system (4) for model B. The first-order terms for model A are U^A, C^A, (and $S^A=0$) and have been published in [2]. Now we proceed to solve the more complicated problem for U^B, C^B, and S^B and to compare these with the previous results.

The expansion $s\sim 0+S^B(x,t)\lambda+\ \ldots$ shows at once that the perturbation procedure for model B cannot be expected to retain validity in the neighborhood of the forward wave front, since $s(x,t,\lambda)=\infty$ for any $\lambda>0$ on the wave front whereas $s(x,t,0)=0$ there. The results therefore will not apply in the neighborhood of the forward wave front where a type of boundary layer effect occurs. A separate analysis would be required in that region, possibly similar to the application of the Pohlhausen method as used by Whitham [7] for a water wave problem. There would be little justification for it in this problem, however, because the model assumed for frictional dissipation would in any case become inaccurate in this region of extremely low density. (A singular situation does not occur in the perturbation for model A. In that case there is no edge layer effect, and results hold throughout the full wave; but on the other hand, the basic model itself is everywhere less accurate than model B.)

When expansions of type (9) are inserted into system (4), after using (10), the first-order terms satisfy the system:

$$\left.\begin{aligned} &\frac{(5t-x)^2}{t^2}S^B_x-\frac{30(t+x)}{t}U^B_x-36U^B_t-\frac{30(5t-x)}{t}C^B_x-\frac{30}{t}U^B+\frac{30}{t}C^B+\frac{25(t+x)^2}{t^2}=0\\ &\frac{(5t-x)}{t}U^B_x+\frac{25(t+x)}{t}C^B_x+30C^B_t-\frac{5}{t}U^B+\frac{5}{t}C^B-\frac{175(t+x)^3}{6t^2(5t-x)}=0\\ &\frac{5(t+x)}{t}S^B_x+6S^B_t-\frac{125(t+x)^3}{t(5t-x)^2}=0. \end{aligned}\right\} \tag{11}$$

The solution of these equations that is sought must satisfy the conditions $U^B=C^B=S^B=0$ on the straight line $t=-x$, $(t\geq 0)$. This nonhomogeneous linear system has the same characteristic curves as the solution u^0, c^0 plus its particle trajectories. By studying these curves one can infer an easy procedure for obtaining the desired functions. These curves in the flow wedge are

$$\left.\begin{aligned} &\zeta_1^0\colon\quad x=m\,t, && -1\leq m\leq 5\\ &\zeta_2^0\colon\quad \left(\frac{x}{a}\right)=5\left(\frac{t}{a}\right)-6\left(\frac{t}{a}\right)^{2/3}, && 0<a\\ &\zeta_3^0\colon\quad \left(\frac{x}{a}\right)=5\left(\frac{t}{a}\right)-6\left(\frac{t}{a}\right)^{5/6}, && 0<a \end{aligned}\right\} \tag{12}$$

where each equation defines a one-parameter family of curves over the range of a parameter as indicated. The ζ_2^0 and ζ_3^0 curves emanate from the back wave line $x=-t$ at a point $(-a,a)$ for any $a>0$ (see fig. 2). Writing ζ_1^0 as $(x/a)=m(t/a)$, one sees that the coordinates of an intersection point of a fixed ζ_1^0 curve with any ζ_2^0 curve must be proportional to the parameter a. Therefore as the parameter a is varied, the quantities t/a and x/a at such intersection points remain constant as we progress along a fixed ζ_1^0 ray. Likewise an analogous situation will

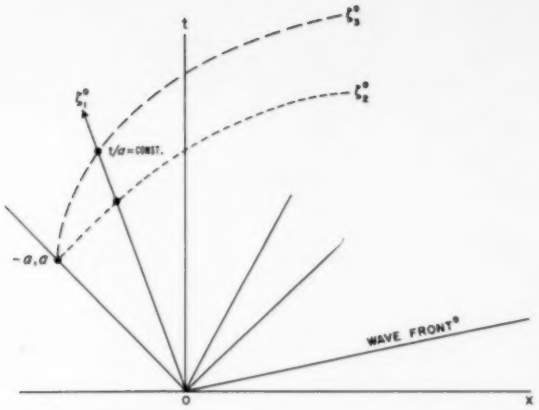

FIGURE 2.

apply for intersections of ζ_3^0 curves with a fixed ζ_1^0 ray. This geometric similarity of the characteristics can be employed to find the functional form of the desired solutions U^B, C^B, S^B. The characteristic relations associated with (12) are

$$\left.\begin{aligned}
&\zeta_1^0\text{:}\quad d(U^B-5C^B)+\left(\frac{5t-x}{6t}\right)dS^B=\left(\frac{25(x+t)^3}{18t^2(x-5t)}-\frac{25(x+t)^2}{36t^2}\right)dt-\frac{5}{3}\frac{U^B-C^B}{t}\,dt\\
&\zeta_2^0\text{:}\quad d(U^B+5C^B)-\left(\frac{5t-x}{6t}\right)dS^B=-\left(\frac{25(x+t)^3}{18t^2(x-5t)}+\frac{25(x+t)^2}{36t^2}\right)dt\\
&\zeta_3^0\text{:}\quad dS^B=\frac{125(t+x)^3}{6t(5t-x)^2}\,dt.
\end{aligned}\right\}\tag{13}$$

The last equation does not contain unknowns on the right side and could therefore be integrated along any ζ_3^0 curve. If this were done, using the third equation in (12) and the geometric similarity, it is easily seen that it would lead to a relation of the form $S^B/a=F_3(t/a)$ when $S^B=0$ at the origin. Because t/a is constant along a ζ_1^0 ray, S^B is proportional to a, and hence to t there; the functional form must then be $S^B(x, t)=L(m)t$. Using this result for dS^B in the second equation of (13), by analogous argument applied to the ζ_2^0 curves, it follows that $U^B+5C^B=F_2(m)t$. Finally, we see that if we integrate the first equation of (13) along any ζ_1^0 ray (where $m=$constant), we would have (U^B-5C^B) proportional to t if the quantity $5(U^B-C^B)/3t$ were a constant along the ray. Then it would follow that U^B and C^B would separately be proportional to t also. But in that case, $5(U^B-C^B)/3t$ would actually be constant as desired. There these considerations indicate the existence of a solution to (13) and (11), vanishing at the origin as required, with the functional form $f(m)t$ for each unknown U^B, C^B, and S^B. This information now permits immediate solution of the original system (11), as follows:

First, system (11) is transformed through use of the new independent variables m and t, putting $U^B(x, t)=U^*(m, t)$, and likewise for the other unknowns. Then setting $U^*=h(m)t$, $C^*=k(m)t$, and $S^*=L(m)t$ leads immediately to a nonhomogeneous system of three ordinary differential equations for h, k, and L. Although this system contains variable coefficients, it can nevertheless be solved exactly. The particular solution satisfying the conditions $h(-1)=k(-1)=L(-1)=0$ is then finally obtained to give the first-order effects for model B.

$$\left.\begin{aligned}
\frac{U^B}{t}&=\frac{25}{48}\left[\frac{35}{108}(5-m)^3-7(5-m)^2+54(5-m)-172+\frac{180}{(5-m)}\right]\\
\frac{C^B}{t}&=\frac{5}{48}\left[\frac{(5-m)^7}{72{,}576}+\frac{35}{54}(5-m)^3-15(5-m)^2+108(5-m)+(1+m)^2-\frac{1868}{7}+\frac{90}{(5-m)}+\frac{7(1+m)^3}{(5-m)}\right]\\
\frac{S^B}{t}&=125\left[\frac{(5-m)^6}{2{,}177{,}280}-\frac{1}{5}(5-m)+3-\frac{108}{7(5-m)}+\frac{27}{(5-m)^2}\right].
\end{aligned}\right\}\tag{14}$$

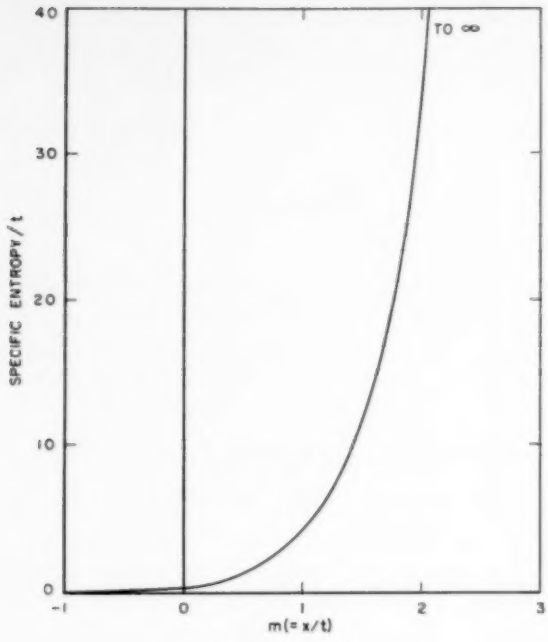

FIGURE 3.

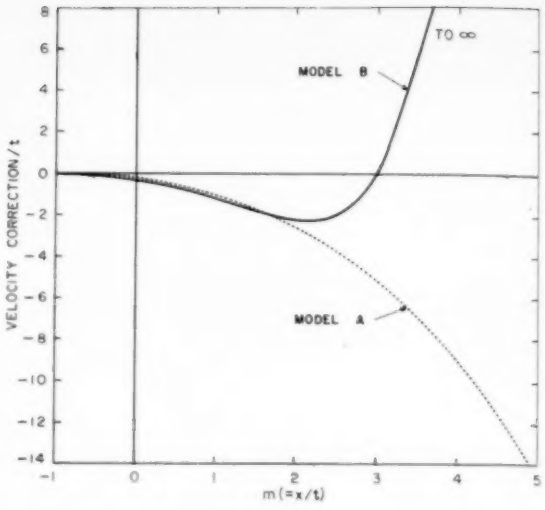

FIGURE 4.

For comparison of these results with the corresponding expressions for the simpler flow model, we list below the expressions obtained in [2] for model A:

$$\left.\begin{aligned} \frac{U^{A}}{t} &= \frac{25}{144}\left[\frac{2}{9}(5-m)^3-5(5-m)^2+36(5-m)-84\right] \\ \frac{C^{A}}{t} &= \frac{5}{144}\left[\frac{4}{9}(5-m)^3-10(5-m)^2+72(5-m)-168+3(m+1)^2\right] \\ \frac{S^{A}}{t} &= 0. \end{aligned}\right\} \tag{15}$$

The expressions in (14) have singularities at $m=5$, at the forward wave front for the unperturbed flow. This is to be expected in view of the previous remarks concerning the entropy behavior at the forward wave front that makes the perturbation procedure singular. On the other hand, the expansion for model A does not possess this singular property, and the expressions (15) do not become infinite at $m=5$. A comparison of (14) with (15) must be restricted to a region that excludes the forward wave-front zone. Figure 3 gives the numerical evaluation of the quantity S^{B}/t. When drawn on a semilogarithmic plot, this curve is almost linear over the middle range $1<m<4$, and for convenience can be accurately approximated there by the expression .63 exp (1.95 m). Figure 4 compares the behavior of U^{B}/t with U^{A}/t. In the first half of the flow zone, the two models predict almost identical results, but further on the velocity becomes higher for model B. In equations (4), as $s_x>0$ for this case, the effect of the entropy term in the momentum equation is to diminish the total resistive force, and this by itself would increase the velocity. In the continuity equation, however, the extra positive term on the right side would have the effect by itself of decreasing the velocity in the back portion of the flow zone, because this term is analogous to the term that would be present for a converging duct in quasi-one-dimensional flow, as derived in [2]. Evidently the fact that the velocity term for model B is slightly below the model A curve in the beginning of the flow zone, is due to the extra term in the continuity equation dominating over the entropy term in the momentum equation until s_x grows sufficiently large to raise the velocity.

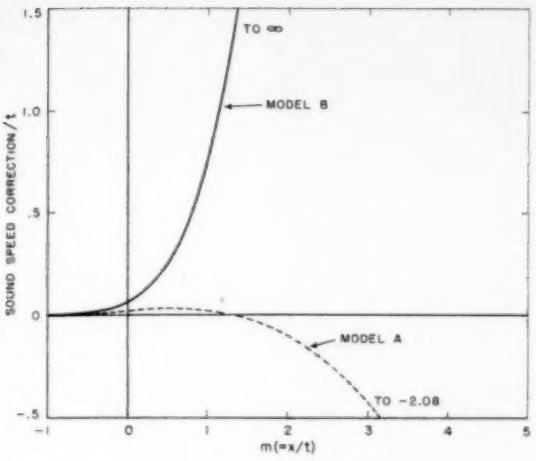

FIGURE 5.

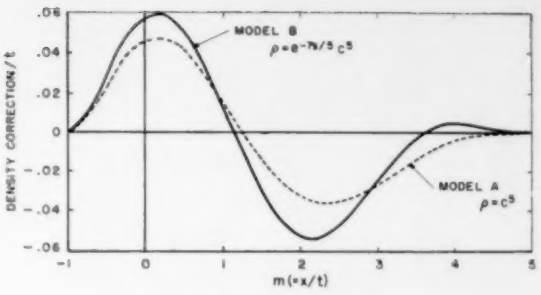

FIGURE 6.

The first-order corrections for the local sound speed c are shown in figure 5 for the two models. In this case there is no resemblance in the behavior of the two curves, as the sound speed, which is proportional to the square root of the temperature, is more directly sensitive than the velocity to the internal heat generation assumed in model B.

Finally we use our results for sound speed and entropy in the two models to calculate the corrections for the dimensionless density $\rho=\overline{\rho}/\overline{\rho}_0$. The dimensionless equation of state of air for model A corresponding to (1) is $\rho=c^5$. After inserting expansions here, the coefficient of λ in the expansion of type (9) for ρ is then equal to $5(c^0)^4C^{\mathrm{A}}$. This quantity, after division by t, is shown in figure 6. For model B, when corresponding expansions are inserted in (8), the coefficient of λ in the ρ expansion is equal to $5(c^0)^4\,[C^{\mathrm{B}}-(7c^0S^{\mathrm{B}}/25)]$. This quantity, after division by t, is likewise shown in figure 6, and one observes that the widely different results of the two models for sound speed and entropy nevertheless combine to produce density corrections that are in close agreement. The pressure and temperature behavior for the wave can be obtained directly from (3) using the results already presented.

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Statistical Investigation of the Fatigue Life of Deep-Groove Ball Bearings

J. Lieblein and M. Zelen

Fatigue is an important factor in determining the service life of ball bearings. Bearing manufacturers are therefore constantly engaged in fatigue-testing operations in order to obtain information relating fatigue life to load and other factors. Several of the larger manufacturers have recently pooled their test data in a cooperative effort to set up uniform and standardized ball-bearing application formulas, which would benefit the many users of antifriction bearings. These data were compiled by the American Standards Association, which subsequently requested that the National Bureau of Standards perform the necessary analyses. This paper summarizes the principal results of the analyses undertaken by the Bureau, and describes the statistical procedures used in the investigation.

1. Introduction

1.1. Statement of Problem

The experience of ball-bearing manufacturers over many years has led to the acceptance of an equation of the form [15, p. 15, eq (53)] [1]

$$L=(C/P)^p, \tag{1}$$

relating fatigue life L to load P when other factors are kept constant. In the above equation, C is termed the "basic (dynamic) capacity," and is defined [15, p. 48] as the constant bearing load (in pounds) that 90 percent of a group of similar bearings can endure for one million revolutions under the given running conditions.

The quantity C in eq (1) depends upon the characteristics of the bearing type, as indicated in [15, p. 32, eq (120)]. When the expression cited is substituted in eq (1), the *fatigue-life formula for ball bearings* takes the form

$$L=\left[\frac{f_c Z^{a_1} D_a^{a_2} (i\cos\alpha)^{a_3}}{P}\right]^p. \tag{2}$$

The symbols are defined as follows:

Z = number of balls.

D_a = ball diameter in inches.

i = number of rows.

α = contact angle.

P = bearing load in pounds.

L = number of million revolutions that a specified percentage of bearings will fail to survive on account of fatigue causes. If the percentage is 10, then $L=L_{10}$, and is termed the *rating life*; if the percentage is 50, then $L=L_{50}$, the *median life*.

p, a_1, a_2, a_3, f_c are taken as unknown parameters whose values have to be estimated from given data.

Since $i=1$ and $\alpha=0°$ for *deep-groove* ball bearings, with which this paper is exclusively concerned, the life equation that will henceforth be considered takes the form

$$L=\left[\frac{f_c Z^{a_1} D_a^{a_2}}{P}\right]^p. \tag{2a}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

The main goal of this investigation was to determine the "best values" of the unknown parameters in the life equation from the experimental data. One of the major problems was to determine the value of the exponent p, as there was disagreement within the ball-bearing industry whether an appropriate value for p was 3, 4, or some other value.

1.2. Description of Data

The data available for analysis consisted of sets of records summarizing endurance tests for deep-groove ball bearings. These tests were carried out over a period of years by four major ball-bearing companies. In the interest of trade anonymity, these companies will henceforth be designated by A, B, C, and D. Each endurance test consisted of a number of bearings of the same type (the number varying from test to test), which were tested simultaneously under the same load and running conditions. Table 1 summarizes the number of test groups of data for each company. The data from company B were sufficiently extensive to permit a further breakdown into three bearing types, here denoted by B–1, B–2, and B–3.

TABLE 1. *Summary of ball-bearing data*

Company	Number of test groups	Total number of bearings in test groups
A	50	1, 259
B	148	3, 289
Type B–1	37	----------
Type B–2	94	----------
Type B–3	17	----------
C	12	291
D	3	109
Total (all companies)	213	4, 948

The worksheets, summarizing the tests, recorded the number of millions of revolutions reached by each bearing in the test group before fatigue failure. Information was also given for those tests terminated before all bearings in the test group failed. In addition to the test results, the worksheets included information on the characteristics of the bearing type (e. g., values for Z, D_a, i, α) and load P, as well as other items of descriptive and identifying information. A specimen worksheet is reproduced in appendix A.

All necessary quantities for evaluating the unknown parameters in the life equation (2) were given directly on the worksheets except the fatigue life L.[2] This quantity can be estimated from the observed fatigue lives of individual bearings within a test group. As already noted, two concepts of fatigue life are used for L, namely, the *rating life* L_{10}, and the *median life* L_{50}. Separate analyses have been carried out with regard to each throughout.

Appendix A summarizes the data taken from the original worksheets that were used in the statistical analysis. Also given are the computed values for L_{10}, L_{50}, and the "Weibull slope" e (which relates to the dispersion of fatigue lives). The methods for obtaining these quantities from the bearing data are given in detail in appendix B.

1.3. Assumptions for the Statistical Analyses

All conclusions reached in this report, and all statistical analyses employed, are based upon the following principal assumptions:

(a) The life formula (2) is the proper functional form for describing fatigue life in ball bearings.

(b) Differences in the measured life of bearings classed as identical, tested at the same load, reflect only the inherent variability of fatigue life, and are free from systematic errors that may arise from different test conditions, materials, manufacturing methods, etc.

[2] Certain estimates for L_{10} and L_{50} had been entered on the worksheets for many of the tests. However, these were not regarded as part of the data submitted for analysis.

(c) All the bearings in a test group can be regarded as a random sample from a homogeneous population of ball bearings.

(d) The probability distribution of the number of revolutions to fatigue failure is of the same form for each test group, although its parameters may differ from group to group.

(e) This fatigue-life distribution is of the type known as the "Weibull distribution."

The purely statistical assumptions, (c) to (e), served as the basis for the determination of L_{10}, L_{50}, and e for each test group. Assumption (e), however, is not involved in the methods used to evaluate the parameters in the life formula (2) from given values of L_{10} or L_{50}. A different assumed form for the distribution of fatigue life might give somewhat different values for L_{10} and L_{50}, but the same methods could then be used to evaluate the unknown parameters in the life formula (2).

Other assumptions of a more technical nature were necessary in the course of the analyses. These are discussed in appendixes B and C.

As in all cases where inferences are made from given data, the conclusions reached here pertain only to the population from which the given data can be regarded as constituting a random sample.

2. Outline of Statistical Analyses

The statistical analyses were divided into two phases. The first phase considered the problem of finding estimates of L_{10}, L_{50}, and the Weibull slope e from the given test data; the second phase used these estimates of L_{10} and L_{50} to evaluate the unknown values of the parameters in the life formula.

2.1. Estimation of L_{10} and L_{50}

The quantity L depends upon the existence of an underlying probability distribution of bearing lives. Selection of a distribution or population is equivalent to specifying the probability that a bearing selected at random from such a population will survive any given number of revolutions, L, or, conversely, that if c is a specified probability, then L is the life period that will be survived with this probability, e. g.,

$$\text{Probability } \{\text{life} \geq L\} = c = \begin{cases} .90 \text{ for } L = L_{10}. \\ .50 \text{ for } L = L_{50}. \end{cases}$$

Accordingly, any L, such as L_{10} or L_{50}, must be obtained by estimating a characteristic of the assumed distribution. For reasons described in appendix B, the distribution characterizing ball-bearing fatigue life was taken to be the Weibull distribution. In brief, this distribution can be derived by assuming a "weakest-link" concept of fatigue strength. In addition, the suitability of the Weibull distribution for fatigue life has been verified in many cases by empirical plotting of data.

One method of estimating L_{10} or L_{50} makes use of special probability-plotting paper so designed that a theoretical Weibull distribution plots as a straight line, and treats the problem as one of straight-line fitting by conventional least squares procedures. However, the procedure usually followed does not take into full account the number of bearings that remain intact when tests are incomplete, nor the interdependence of successive points. Because of these and other limitations, it was decided to use an alternative approach in the estimation of L_{10} and L_{50} for each test group (see appendix B).

To this end, a method was developed that takes into account explicitly the number of bearings remaining intact at the termination of a test, and that also possesses several other advantages. This method makes use of certain specially determined linear functions of the observed failure times (in logarithms), x_i, arranged in order of size. These functions have the general form

$$T = \sum_{j=1}^{k} c_j x_j. \tag{3}$$

As the method makes intimate use of the ordered arrangement of the data, it is termed an "order statistics" method.

The coefficients c_j in eq (3) allow great flexibility. They have been determined in such a manner that the method will have certain desirable objective characteristics, e. g., freedom from systematic error and a minimum standard error.

2.2. Evaluation of the Parameters in the Life Formula

Once the estimates for L_{10} and L_{50} are obtained, it is possible to evaluate the exponent p in the life formula. However, in order to make the most efficient use of the given data, it is necessary also to estimate the other parameters, f_c, a_1, and a_2.

The methods for estimating the values of L_{10} and L_{50} for each test group actually yield results for ln L_{10} and ln L_{50}. Thus, taking logarithms [3] of the life equation (2a) gives

$$\ln L=(p \ln f_c)+(pa_1) \ln Z+(pa_2) \ln D_a-p \ln P. \tag{4}$$

This equation can be written more simply as

$$Y=b_0+b_1x_1+b_2x_2+b_3x_3, \tag{5}$$

where

$$\left.\begin{aligned} &Y=\ln L \text{ (for either } L_{10} \text{ or } L_{50}),\\ &b_0=p \ln f_c,\ b_1=pa_1,\ b_2=pa_2,\ b_3=-p,\\ &x_1=\ln Z,\ x_2=\ln D_a,\ x_3=\ln P. \end{aligned}\right\} \tag{6}$$

The quantities x_1, x_2, and x_3 depend on the characteristics of the bearing type and test conditions, and can be regarded as known exactly. On the other hand, the variable Y, which depends on the outcome of the bearing tests, is subject to considerable dispersion. Thus, estimates can be found for the parameters b_0, b_1, b_2, and b_3, using standard least squares methods based on minimizing the sums of squared deviations in the y direction. These methods are discussed in detail in appendix C.

After the parameters b_0, b_1, b_2, and b_3 are estimated, values for a_0, a_1, a_2, and p can be found from the relations

$$a_0=\ln f_c=-\frac{b_0}{b_3} \qquad a_1=-\frac{b_1}{b_3}$$

$$a_2=-\frac{b_2}{b_3} \qquad p=-b_3$$

It is clear that the values for a_0, a_1, and a_2 depend on the value of p.

The estimates for p and the a's are subject to some uncertainties because they are based on test results, which themselves are subject to considerable variability. Hence with every value of p and of the a's calculated from the life data, there is given also an interval of uncertainty to indicate its precision. These intervals are "95-percent confidence limits." [4]

A large interval of uncertainty associated with an estimate indicates poor precision; a small interval of uncertainty is evidence of high precision. These intervals of uncertainty not only reflect the inherent variability of the test data, but are also affected by (a) how well the life equation (2a) is the proper functional form for bearing life, and (b) the suitability of the data (including the number of test groups) for estimating the parameters in the life formula.

Further technical details concerning the evaluation of the parameters in the life formula are given in appendix C.

[3] Natural logarithms to the base e are used throughout.

[4] Briefly, confidence intervals describe the compatibility of the observations with an unknown parameter estimated from them; 95-percent confidence limits are limits such that on the average, in repeated applications of the same procedure, 95 percent of intervals so calculated will contain the unknown true value of the parameter. The confidence limits associated with p are symmetric. However, the confidence limits associated with the a's are asymmetric because of the dependence of the a's on p.

3. Summary of Analyses

3.1. Evaluation of Parameter p

The statistical analysis based on all deep-groove ball-bearing data from companies A, B, and C [5] yielded the final values for p shown in table 2. The separate values for each of the three companies are given in table 3. The intervals of uncertainty specified by $\pm$ quantities in those tables refer to intervals within which, with reasonable assurance, the true value of the parameter is located. The fact that all of the intervals of uncertainty exhibit considerable overlap shows that the data are consistent with the supposition that all three companies have a common value of p for deep-groove bearings. The fact that all the intervals include 3 indicates that all the estimates of p are consistent with the practice of taking $p=3$. Moreover, the value of p for L_{10} was not significantly different from that for L_{50}.

TABLE 2. *Final over-all values of p for deep-groove bearings*

	L_{10}	L_{50}
p	2.87 ± 0.35	2.80 ± 0.31

TABLE 3. *Individual estimates of p for deep-groove bearings by company*

Company	Number of test groups	L_{10}	L_{50}
A	50	3.00 ± 0.64	3.05 ± 0.60
B	148	$2.75 \pm .48$	2.62 ± 0.40
C	12	$3.12 \pm .88$	2.88 ± 1.02

The values given for p are based on analyses of all deep-groove ball-bearing data, irrespective of bearing type. Hence, the parameter estimates represent "omnibus" values. In order to investigate the dependence of the exponent p on bearing type, the data from company B, which was made up of three bearing types, were analyzed separately. The results for the exponent p are shown in table 4. These results are all compatible with the value $p=3$.

TABLE 4. *Value of p by bearing type for company B*

Type	Number of test groups	Value of p	
		L_{10}	L_{50}
B–1	37	3.36 ± 0.68	3.23 ± 0.47
B–2	94	2.65 ± 0.91	2.13 ± 0.79
B–3	17	1.89 ± 1.28	2.82 ± 1.10
Total	148		

[5] The data furnished by company D were too few to be included in the analysis.

3.2. Evaluation of Parameters f_c, a_1, and a_2

The computations that give estimates for the exponent p also yield estimates for the quantities $\ln f_c$, a_1, and a_2. From the relations (6) it is clear that the values for these parameters depend on the value for p. Thus, associated with every value of p will be corresponding values for $\ln f_c$, a_1, and a_2. Table 5 summarizes these parameter estimates associated with the final values of p. The estimates for $a_0 = \ln f_c$, rather than f_c, are given here, because this is the parameter that arises naturally in the life formula (cf. eq 4).

The analyses conducted separately for each company resulted in other values than those in the previous paragraph for a_0, a_1, a_2. These results are summarized in table 6. They show excellent agreement with the results in table 5, even though the values for p are somewhat different.

TABLE 5. *Final values of a_0, a_1, a_2 for L_{10} and L_{50}*

Company	p	a_0	Interval of uncertainty	a_1	Interval of uncertainty	a_2	Interval of uncertainty
			L_{10}				
A	2. 87	9. 02	(7. 31, 10. 79)	0. 380	(−0. 454, 1. 201)	1. 72	(1. 51, 1. 92)
B	2. 87	8. 55	(7. 98, 9. 14)	. 670	(0. 418, 0. 920)	1. 81	(1. 70, 1. 92)
C	2. 87	9. 56	(6. 85, 12. 42)	−. 174	(−1. 750, 1. 352)	1. 37	(0. 09, 2. 67)
			L_{50}				
A	2. 80	10. 36	(8. 81, 11. 98)	0. 015	(−0. 741, 0. 751)	1. 69	(1. 50, 1. 88)
B	2. 80	9. 05	(8. 54, 9. 60)	. 695	(. 470, . 920)	1. 91	(1. 81, 2. 01)
C	2. 80	9. 05	(6. 61, 11. 58)	. 475	(− . 921, 1. 847)	1. 76	(0. 60, 2. 93)

TABLE 6. *Values of a_0, a_1, a_2 for L_{10} and L_{50}, based on independent analyses for each company*

Company	p	a_0	Interval of uncertainty	a_1	Interval of uncertainty	a_2	Interval of uncertainty
			L_{10}				
A	3. 00	8. 97	(7. 18, 10. 90)	0. 390	(−0. 507, 1. 249)	1. 73	(1. 50, 1. 94)
B	2. 75	8. 59	(7. 99, 9. 24)	. 666	(. 398, 0. 928)	1. 80	(1. 67, 1. 92)
C	3. 12	9. 21	(7. 29, 11. 84)	−. 041	(−1. 326, . 992)	1. 36	(0. 49, 2. 30)
			L_{50}				
A	3. 05	10. 13	(8. 48, 12. 00)	0. 072	(−0. 768, 0. 855)	1. 71	(1. 50, 1. 91)
B	2. 62	9. 15	(8. 61, 9. 76)	. 690	(. 456, . 922)	1. 90	(1. 79, 2. 00)
C	2. 88	8. 93	(6. 58, 12. 39)	. 510	(−1. 055, 1. 810)	1. 75	(0. 66, 3. 05)

Similarly, the values for a_0, a_1, and a_2, arising from separate analyses made on the three types of bearings from company B, resulted in still other estimates for these parameters. Table 7 summarizes these estimates. These estimates are less precise than the corresponding omnibus values given for company B in table 6. This is a consequence of the fact that within a bearing type, the quantities Z and D_a hardly vary at all. This condition makes the data unsuitable for estimating the associated unknown parameters, a_0, a_1, and a_2.

TABLE 7. *Values of a_0, a_1, a_2 for L_{10} and L_{50} by bearing type (company B)*

Type	a_0	Interval of uncertainty	a_1	Interval of uncertainty	a_2	Interval of uncertainty
			L_{10}			
B–1	7. 25	(5. 21, 9. 39)	1. 07	(0. 35, 1. 76)	1. 68	(1. 27, 2. 08)
B–2	7. 34	(5. 54, 9. 33)	1. 21	(.43, 2. 00)	1. 69	(1. 38, 1. 93)
B–3	2. 50	(−7. 95, 16. 04)	3. 70	(−2. 25, 9. 06)	1. 27	(0. 30, 1. 65)
			L_{50}			
B–1	7. 39	(5. 92, 8. 92)	1. 23	(0. 73, 1. 73)	1. 79	(1. 50, 2. 08)
B–2	9. 00	(7. 10, 11. 67)	0. 87	(−. 05, 1. 68)	1. 77	(1. 46, 2. 03)
B–3	1. 03	(−4. 34, 6. 53)	4. 50	(1. 97, 7. 19)	1. 48	(1. 21, 1. 70)

3.3. Redetermination of the Estimates for f_c

The uncertainty intervals associated with estimates for the parameter $a_0=\ln f_c$ are quite large. This is primarily because the uncertainty associated with the estimate of a_0 also depends on how well the other parameters, a_1, a_2, and p, are estimated. Another way to evaluate a_0, which may result in smaller intervals of uncertainty, is to assume a priori values for a_1, a_2, and p, and then determine the estimate for a_0. This procedure was followed by using the widely accepted values for the parameters given in [15], namely, $a_1=2/3$, $a_2=1.8$, $p=3$.

However, if on such a calculation the values assumed for the parameters a_1, a_2, and p are not compatible with the given data, then values of a_0 (or f_c) so calculated will not be correct determinations for these data. Accordingly, an analysis was made to determine whether the parameter values in [15] were compatible with the given data.

This analysis showed that these parameter values are compatible with the data, with respect to all individual companies for rating life L_{10}, but not for median life L_{50}. (Company A was the only company for which the parameter values are suitable for median life.) A further analysis, by bearing type for company B, showed that the above parameter values are not suitable for the rating life L_{10} with respect to B–3-type bearings.

In the light of this last analysis, redetermined values of a_0, taking $a_1=2/3$, $a_2=1.8$, and $p=3$, are only strictly valid with respect to company A, company B (B–1, B–2), and company C for rating life L_{10}. These values are summarized in table 8. For convenience, these new estimates are given for $f_c=\ln^{-1}a_0=\exp a_0$.

TABLE 8. *Values for f_c assuming $a_1=2/3$, $a_2=1.8$, $p=3.0$ for L_{10}*

Company	Number of test groups	f_c	Interval of uncertainty
A	50	4, 538	(4, 273, 4, 817)
B (over-all value)	148	4, 925	(4, 750, 5, 105)
B–1	37	4, 709	(4, 403, 5, 034)
B–2	94	5, 033	(4, 885, 5, 187)
B–3*	17	-----	----------
C	12	3, 294	(3, 029, 3, 583)
D	3	4, 639	(3, 478, 6, 187)

*Assumed values of parameters a_1, a_2, and p not compatible with test results for bearings of this series.

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4. Appendix A. Summary of Original Data

This appendix summarizes in tabular form the worksheets submitted by the American Standards Association Subcommittee to the National Bureau of Standards for statistical analysis. Separate tables are presented for deep-groove data from companies A, B, C, and D. These four tables (A–1 to A–4) are followed by table A–5, which gives a synopsis of the number of test groups and the number of bearings for each company.

Tables A–1 to A–4 give the size of test group, the values for quantities P, Z, D_a, and the estimates [6] for L_{10}, L_{50}, and the "Weibull slope" e. All of these variables are directly observed or specified quantities except for the estimates L_{10}, L_{50}, and e. These last three quantities are based on statistical calculations that made use of the results of individual endurance tests. These calculations are explained in appendix B.

The original data, as submitted, contained a few cases where companies tested bearings manufactured by other companies. Such test groups are not included in the summary tables, as these results confound differences in testing with differences in manufacturing. Therefore these test results were not used in any of the analyses. Thus, table A–3, for company C, omits 4 tests performed on other manufacturers' bearings; table A–4, for company D, omits 3 tests.

The five tables described above are followed by a specimen worksheet [7] with identifying information removed. A sample of Weibull-function coordinate paper is also included. This coordinate paper had been used for graphing the results of all the individual endurance tests and these graphs had accompanied the worksheets submitted to the Statistical Engineering Laboratory.

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[6] The estimates for L_{10} and L_{50} are given in millions of revolutions for all companies except company D. The life estimates for D are shown in hours, the same units in which the original endurance data were given.

[7] Bearings marked "Omitted" were completely eliminated from consideration, as company representatives explained that these were non-fatigue failures and should not be regarded as part of the test group. As a result, the test group in the case of the specimen sheet shown was taken to consist of 23 bearings rather than the original number of 25. This type of situation appeared rather infrequently, however.

TABLE A–1. *Summary ball-bearing data for company A, with computed values for L_{10}, L_{50}, and Weibull slope e*

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}	L_{50}	e Weibull slope
			lb		in.			
1- 1	1936	24	4240	8	11/16	19.2	84.5	1.27
1- 2	1937	20	4240	8	11/16	26.2	74.2	1.81
1- 3	1937	14	4240	8	11/16	11.1	68.1	1.04
1- 4	1937	19	4240	8	11/16	11.8	66.8	1.09
1- 5	1937	18	4240	8	11/16	13.5	79.4	1.06
1- 6	1938	21	2530	9	1/2	5.80	25.7	1.27
1- 7	1938	28	4240	8	11/16	18.3	44.7	2.10
1- 8	1938	27	4240	8	11/16	5.62	73.2	.73
1- 9	1940	20	4240	8	11/16	15.8	82.7	1.14
1-10	1940	22	4240	8	11/16	8.70	41.6	1.20
1-11	1940	19	4240	8	11/16	11.6	160	.72
1-12	1940	15	1940	9	7/16	20.6	71.4	1.52
1-13	1940	15	1940	9	7/16	14.5	88.2	1.04
1-14	1940	15	2536	9	1/2	12.1	33.1	1.87
1-15	1940	14	2536	9	1/2	15.1	46.4	1.67
1-16	1940	15	2536	9	1/2	14.0	43.6	1.66
1-17	1940	14	2536	9	1/2	19.3	51.8	1.91
1-18	1940	26	4240	8	11/16	46.2	110	2.17
1-19	1940	14	4240	8	11/16	30.0	88.2	1.74
1-20	1942	20	4240	8	11/16	21.1	57.4	1.89
1-21	1942	20	4240	8	11/16	17.3	45.7	1.94
1-22	1942	37	4240	8	11/16	37.5	118	1.64
1-23	1942	36	4240	8	11/16	20.3	77.1	1.41
1-24	1942	32	4240	8	11/16	4.03	42.5	.80
1-25	1941	28	2544	8	17/32	8.38	84.7	.81
1-26	1943	23	3975	8	19/32	1.79	13.5	.93
1-27	1942	30	4400	10	5/8	11.7	45.1	1.39
1-28	1942	31	6920	8	7/8	4.15	15.8	1.41
1-29	1943	30	990	9	5/16	7.23	41.0	1.09
1-30	1943	30	1509	7	7/16	22.9	110	1.20
1-31	1943	30	932	7	11/32	9.54	31.6	1.57
1-32	1944	26	3180	8	19/32	6.28	23.0	1.45
1-33	1944	29	3180	8	19/32	4.81	21.2	1.27
1-34	1944	33	8640	10	7/8	4.17	12.8	1.68
1-35	1944	26	14080	8	1-1/4	5.42	31.6	1.07
1-36	1951	28	1940	9	7/16	7.47	49.5	1.00
1-37	1951	34	2330	9	7/16	4.80	21.3	1.26
1-38	1951	27	1550	9	7/16	14.8	78.4	1.13
1-39	1951	29	1165	9	7/16	84.9	460	1.11
1-40	1951	27	2910	9	7/16	3.40	16.5	1.19
1-41	1951	27	3880	9	7/16	1.24	3.23	1.97
1-42	1951	26	776	9	7/16	241	951	1.37
1-43	1951	30	19750	8	1-3/4	3.01	12.6	1.31
1-44		30	2112	8	11/16	89.1	486	1.11
1-45		30	4224	8	11/16	15.2	104	.98
1-46	Not given	30	8448	8	11/16	2.04	10.2	1.17
1-47		30	2112	8	5/8	51.0	376	.94
1-48		30	4224	8	5/8	5.26	58.8	.78
1-49		30	8448	8	5/8	.883	4.94	1.09
1-50	1944	30	4224	8	11/16	14.8	57.4	1.39

TABLE A–2. *Summary ball-bearing data for company B, with computed values for L_{10}, L_{50}, and Weibull slope e*

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}	L_{50}	e Weibull slope
			lb		in.			
2- 1	1940	19	570	10	3/16	6.68	13.4	2.72
2- 2	1944	20	570	9	5/16	29.8	70.0	2.22
2- 3	1946	23	580	9	1/4	16.3	55.1	1.55
2- 4	1946	23	580	9	1/4	28.5	69.2	2.13
2- 5	1947	23	580	9	1/4	16.4	49.3	1.71
2- 6	1943	10	665	9	1/4	10.3	40.1	1.50
2- 7	1944	10	665	9	1/4	25.7	46.4	3.19
2- 8	1942	19	580	10	1/4	9.55	39.6	1.32
2- 9	1946	33	620	10	1/4	17.9	62.1	1.51
2-10	1947	15	620	10	1/4	19.9	73.2	1.45
2-11	1947	31	620	10	1/4	12.9	50.4	1.39
2-12	1944	19	625	10	1/4	19.3	46.2	2.18
2-13	1941	17	720	10	1/4	11.1	23.3	2.54
2-14	1946	60	980	11	9/32	15.7	43.5	1.85
2-15	1947	32	980	11	9/32	11.2	38.1	1.54
2-16	1950	49	600	11	5/16	417	809	2.85
2-17	1949	60	600	11	5/16	216	709	1.58
2-18	1943	20	900	11	5/16	35.6	100	1.82
2-19	1946	67	1220	11	5/16	12.0	42.2	1.50
2-20	1947	34	1220	11	5/16	8.53	46.6	1.11
2-21	1940	20	1370	11	5/16	6.77	18.9	1.85
2-22	1950	60	1415	11	5/16	13.5	46.5	1.53
2-23	1950	60	2243	11	5/16	2.32	8.06	1.51
2-24	1942	20	720	12	5/16	36.7	141	1.40
2-25	1946	55	1300	12	5/16	19.0	57.2	1.71
2-26	1947	30	1300	12	5/16	19.5	60.6	1.67
2-27	1944	20	1650	14	11/32	17.0	74.4	1.37
2-28	1946	59	1760	14	11/32	20.9	53.7	2.00
2-29	1947	34	1760	14	11/32	9.56	40.7	1.30
2-30	1940	20	2010	13	13/32	5.49	33.3	1.05
2-31	1940	9	2010	13	13/32	1.39	44.0	.54
2-32	1944	19	2140	13	13/32	9.80	82.7	.88
2-33	1943	11	2630	15	13/32	5.19	54.9	.80
2-34	1942	12	5900	14	19/32	6.36	17.5	1.86
2-35	1947	19	5900	14	19/32	3.68	22.1	1.05
2-36	1947	20	8070	14	23/32	8.34	23.6	1.81
2-37	1942	12	8075	14	23/32	6.78	36.4	1.12
2-38	1938	10	565	9	.210	9.27	18.4	2.75
2-39	1940	23	720	8	9/32	18.2	56.9	1.66
2-40	1940	24	720	8	9/32	22.8	56.2	2.09
2-41	1940	25	720	8	9/32	3.99	15.6	1.38
2-42	1940	21	720	8	9/32	9.07	29.4	1.60
2-43	1940	25	720	8	9/32	7.14	28.5	1.36
2-44	1940	25	720	8	9/32	12.5	26.4	2.51
2-45	1941	25	720	8	9/32	18.8	48.7	1.98
2-46	1941	23	720	8	9/32	21.5	53.2	2.08
2-47	1947	33	860	8	5/16	17.1	59.0	1.52
2-48	1948	8	860	8	5/16	15.2	87.6	1.08
2-49	1943	20	900	9	5/16	30.1	92.3	1.68
2-50	1944	18	900	9	5/16	15.0	47.6	1.63

TABLE A–2. *Summary ball-bearing data for company B, with computed values for L_{10}, L_{50}, and Weibull slope e*—Continued

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}	L_{50}	e Weibull slope
			lb		in.			
2-51	1945	27	940	9	5/16	17.5	52.8	1.71
2-52	1947	34	940	9	5/16	14.4	65.6	1.24
2-53	1938	10	1180	9	5/16	8.76	22.1	2.04
2-54	1945	30	1580	9	3/8	12.1	43.3	1.47
2-55	1947	33	1580	9	3/8	17.2	64.6	1.42
2-56	1948	8	1580	9	3/8	10.7	34.6	1.61
2-57	1945	31	2160	9	7/16	10.9	37.6	1.52
2-58	1947	30	2160	9	7/16	12.7	53.7	1.30
2-59	1938	9	2200	9	7/16	3.73	43.5	.77
2-60	1947	30	2480	9	7/16	16.6	78.3	1.21
2-61	1950	40	1340	9	15/32	180	275	4.44
2-62	1937	19	1660	10	7/16	85.2	234	1.86
2-63	1941	19	1700	9	15/32	57.1	230	1.35
2-64	1939	24	2480	9	15/32	15.7	55.8	1.48
2-65	1939	25	2480	9	15/32	27.1	97.8	1.47
2-66	1939	23	2480	9	15/32	21.7	122	1.09
2-67	1939	28	2480	9	15/32	13.2	42.3	1.62
2-68	1939	28	2480	9	15/32	35.8	145	1.35
2-69	1939	20	2480	9	15/32	12.7	34.7	1.87
2-70	1944	20	2480	9	15/32	10.1	27.8	1.87
2-71	1945	20	2480	9	15/32	8.83	34.3	1.39
2-72	1938	10	2480	9	15/32	16.5	60.3	1.45
2-73	1942	11	2480	9	15/32	17.9	65.8	1.45
2-74	1943	10	2480	9	15/32	15.7	63.1	1.35
2-75	1943	20	2480	9	15/32	10.8	42.1	1.38
2-76	1944	18	2480	9	15/32	14.2	39.9	1.83
2-77	1944	18	2480	9	15/32	19.0	67.8	1.48
2-78	1944	18	2480	9	15/32	16.3	57.7	1.49
2-79	1944	20	2480	9	15/32	2.93	18.0	1.04
2-80	1944	20	2480	9	15/32	5.69	25.4	1.26
2-81	1944	28	2480	9	15/32	9.54	39.9	1.32
2-82	1944	22	2480	9	15/32	12.6	55.7	1.27
2-83	1944	23	2480	9	15/32	5.10	37.5	.94
2-84	1944	18	2480	9	15/32	16.0	53.7	1.56
2-85	1944	20	2480	9	15/32	1.98	22.1	.78
2-86	1945	20	2480	9	15/32	5.65	28.8	1.16
2-87	1945	20	2480	9	15/32	12.8	43.6	1.58
2-88	1945	20	2480	9	15/32	9.84	32.3	1.59
2-89	1945	20	2480	9	15/32	12.1	43.0	1.48
2-90	1945	20	2480	9	15/32	5.48	40.8	.94
2-91	1945	20	2480	9	15/32	6.64	25.3	1.41
2-92	1945	32	2480	9	15/32	13.9	41.9	1.70
2-93	1946	35	2480	9	15/32	9.02	45.4	1.17
2-94	1946	34	2480	9	15/32	11.0	49.2	1.26
2-95	1947	31	2480	9	15/32	14.5	73.6	1.16
2-96	1944	9	2480	9	15/32	5.91	37.2	1.02
2-97	1944	10	2480	9	15/32	18.1	40.5	2.33
2-98	1945	10	2480	9	15/32	17.1	53.3	1.65
2-99	1945	10	2480	9	15/32	32.6	61.8	2.95
2-100	1945	10	2480	9	15/32	24.1	66.2	1.87

404091—56——3

TABLE A–2. *Summary ball-bearing data for company B, with computed values for L_{10}, L_{50}, and Weibull slope e*—Continued

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}	L_{50}	e Weibull slope
			lb		in.			
2-101	1945	20	2480	9	15/32	36.1	71.6	2.75
2-102	1946	20	2480	9	15/32	63.3	104	3.82
2-103	1946	12	2480	9	15/32	14.4	59.0	1.33
2-104	1946	11	2480	9	15/32	15.1	92.9	1.04
2-105	1945	10	2480	9	15/32	18.8	39.4	2.55
2-106	1950	12	2480	9	15/32	5.63	34.7	1.04
2-107	1950	12	2480	9	15/32	7.23	34.5	1.21
2-108	1951	30	2480	9	15/32	16.7	71.8	1.29
2-109	1951	63	2480	9	15/32	26.5	90.3	1.54
2-110	1950	23	2480	9	15/32	8.35	49.1	1.06
2-111	1943	19	3250	9	15/32	3.79	9.30	2.10
2-112	1937	10	3470	10	7/16	9.05	36.6	1.35
2-113	1944	20	4000	9	15/32	2.98	7.35	2.08
2-114	1943	19	2300	10	15/32	22.5	73.4	1.59
2-115	1938	10	2730	10	15/32	3.82	31.7	.89
2-116	1946	22	2660	10	17/32	6.55	20.8	1.63
2-117	1944	20	2250	11	15/32	17.5	64.3	1.45
2-118	1943	16	2300	11	15/32	61.7	152	2.10
2-119	1945	48	2840	11	15/32	18.6	42.7	2.27
2-120	1947	28	2840	11	15/32	21.6	66.3	1.68
2-121	1947	8	2840	11	15/32	11.9	39.1	1.59
2-122	1948	8	2840	11	15/32	13.9	50.6	1.46
2-123	1943	19	3200	11	15/32	7.80	33.1	1.30
2-124	1944	28	4000	11	15/32	3.55	13.9	1.38
2-125	1943	19	4000	11	15/32	9.40	23.4	2.06
2-126	1947	23	6350	11	11/16	4.76	22.7	1.21
2-127	1944	20	12000	11	1-1/16	3.23	9.86	1.69
2-128	1944	20	12000	11	1-1/16	2.62	9.52	1.46
2-129	1944	9	12700	8	1-1/2	7.89	39.7	1.17
2-130	1949	18	16500	11	1-1/16	4.93	20.4	1.33
2-131	1950	20	16500	11	1-1/16	6.26	16.2	1.98
2-132	1938	8	565	7	5/16	37.3	103	1.85
2-133	1944	20	900	7	5/16	14.0	38.6	1.86
2-134	1938	10	1650	8	13/32	30.3	87.6	1.77
2-135	1944	20	2250	8	15/32	25.7	71.2	1.85
2-136	1943	20	2300	8	15/32	10.5	60.4	1.07
2-137	1944	19	3200	8	15/32	10.3	24.1	2.21
2-138	1944	19	4000	8	15/32	4.56	12.9	1.81
2-139	1937	10	1710	8	17/32	25.1	274	.79
2-140	1938	9	2360	8	17/32	48.8	264	1.12
2-141	1937	10	2680	8	17/32	7.53	60.7	.90
2-142	1937	11	3850	8	17/32	14.9	62.6	1.32
2-143	1947	21	7760	8	29/32	4.57	43.4	.84
2-144	1943	12	9550	8	1-1/16	3.90	40.7	.80
2-145	1947	21	9750	8	1-1/16	15.5	79.4	1.16
2-146	1948	16	11400	8	1-3/16	10.2	43.9	1.29
2-147	1948	20	11400	8	1-3/16	4.71	16.9	1.48
2-148	1947	18	11420	8	1-3/16	10.1	34.2	1.55

TABLE A–3. *Summary ball-bearing data for company C, with computed values for L_{10}, L_{50}, and Weibull slope e*

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}	L_{50}	e Weibull slope
			lb		in.			
3- 1	1942	94	1580	7	9/16	16.9	64.8	1.40
3- 2	1949	29	790	7	9/16	211	729	1.52
3- 3	1949	35	1185	7	9/16	74.4	287	1.40
3- 4	1940	29	1600	9	1/2	9.62	40.1	1.32
3- 5	1945	10	1600	8	15/32	11.9	66.3	1.10
3- 6	1943	9	2275	7	17/32	13.8	58.0	1.31
3- 7	1946	13	2540	8	15/32	2.38	11.3	1.21
3- 8	1946	12	2540	8	15/32	2.38	11.5	1.19
3- 9	1949	12	1580	7	9/16	8.75	62.2	0.96
3-10	1949	12	1580	7	9/16	25.7	113	1.27
3-11	1947	24	1600	9	1/2	14.5	113	0.92
3-12	1949	12	610	8	5/16	26.8	65.6	2.10

TABLE A–4. *Summary ball-bearing data for company D, with computed values for L_{10}, L_{50}, and Weibull slope e*

Record No.	Year of test	Number in test group	Load	Z Number of balls	D_a Ball diam.	L_{10}^*	L_{50}^*	e Weibull slope
			lb		in.			
4-1	1946	19	1750	9	7/16	159	963	1.05
4-2	1951	34	1750	9	7/16	71.7	526	0.94
4-3	1951	56	1750	9	7/16	113	582	1.15

*Life estimates are in hours.

TABLE A–5. *Summary of test groups of ball-bearing data*

Company	Number of test groups	Total number of bearings in test group
A	50	1, 259
B	148	3, 289
Type B–1	37	------
Type B–2	94	------
Type B–3	17	------
C	12	291
D*	3	109
Total (all companies)	213	4, 948

*These data were not used in the main analyses.

SPECIMEN WORKSHEET

Reference No. _____
Bearing Mfg. by _____
Bearing Tested by _____
Date of Test 8-26-46
Bearing No. _____
Load 500 R.L.
Speed 2000 r.p.m.
Lubrication: Type Jet Oil
Frequency _____
Ball No. and Dia. 9 - 1/4"
Contact Angle 0°
Groove Radius: Inner Ring 51.6%
Outer Ring 53.0%
Number of Rows 1
Bore 20 mm.
O.D. 42 mm.
Lot Size 25 Taken on 23

Bearing temperature measured on outer ring at point of maximum load _____
Material: Type _____
Source _____
Rockwell Hardness of:
Inner Ring 63.5
Outer Ring 64.
Balls _____

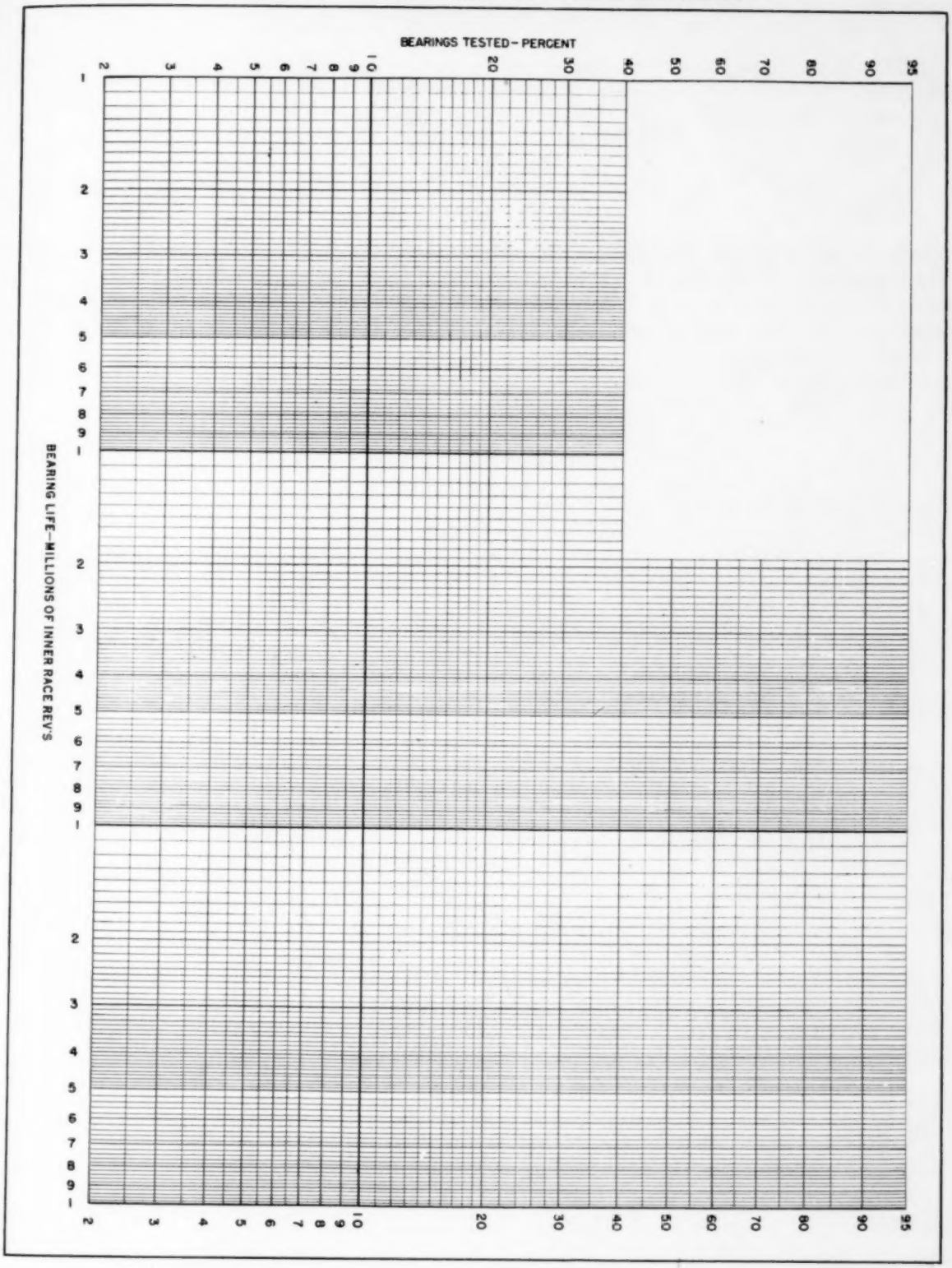
Ball Failure	13	52%
Inner Ring Failure	5	20%
Outer Ring Failure	1	4%

Test life in 10^6 revolutions:
Median 68.
Mean 71.
B-10 29.
Slope of Curve 2.23
Test No. 3183
Lot 71

Table Ordered According to Endurance Life

Brg. No.	Endurance Mill. Revs.	Type of Failure	Remarks
16	17.88	Ball	
10	28.92	Ball	
5	33.00	Ball	
19	41.52	I.R.	
9	42.12	Ball	
11	45.60	Ball	
15	48.48	Ball	
12	51.84	Ball	
20	51.96	Ball	
18	54.12	I.R.	
13	55.56	I.R.	
1	67.80	Ball	
~~2~~	~~67.80~~	~~I.Bore~~	~~Omitted~~
~~3~~	~~67.80~~	~~I.Bore~~	~~Omitted~~
4	68.64	Ball	
6	68.64	L.Bore	
25	68.88 →	Disc.	
22	84.12	Ball	
17	93.12	Ball	
7	98.64	I.R.	
23	105.12	I.R.	
24	105.84 →	Disc.	
21	127.92	Ball	
8	128.04	O.R.	
14	173.40 →	Disc.	

SAMPLE OF WEIBULL FUNCTION COORDINATE PAPER



5. Appendix B. Evaluation of L_{10}, L_{50}, and Weibull Slope e, by Using Order Statistics for Censored Data

This is a technical appendix that gives the mathematical basis for estimating, for each test group, the values of L_{10} and L_{50} for use in the regression analysis discussed in appendix C, and also the Weibull slope e.

5.1. Weibull Distribution

a. Characteristics

As noted in the text, the basic assumption for estimating L_{10}, L_{50}, and e for each test group was that the probability distribution of fatigue lives of individual bearings could be represented by a "Weibull distribution." [8] This means that the observed fatigue lives (number of revolutions) of all the bearings in a test group of, say, n bearings constitute a random sample of n independent observations from a distribution whose cumulative (from above) distribution function (hereafter denoted by cdf) is [9]

$$S(L)=\text{Prob}\{\text{life}\geq L\}$$
$$=\exp\left[-(L/a)^e\right], \qquad 0\leq L<\infty, \tag{B1}$$

where a and e are the two parameters to be fitted. They are related to L_{10} and L_{50} by eq (B2a) below. The function $S(L)$ is also termed the "survivorship" function. This distribution is one of three limiting types to which the distribution of the smallest member of a sample, under general conditions, tends as the sample size is increased indefinitely. (Another type is discussed in the following section.) This matter was first studied chiefly by Fisher and Tippett [5], and for this reason the type (B1) is sometimes referred to as Fisher-Tippett type III for smallest values.

There are both theoretical and practical reasons for choosing the Weibull distribution (B1) as the underlying probability distribution for fatigue life.

Theoretical. Here it is assumed that fatigue is an "extreme-value" phenomenon, related in some manner to the strength at the weakest point in the material under stress. The theoretical reasoning that proceeds from this assumption is mentioned by a number of authors, and is given explicitly, for example, by Freudenthal and Gumbel in [6, p. 316 to 318]. It leads precisely to the form (B1) (see eq (2.9) in [6]). It is recognized that this statistical assumption has not received universal acceptance. This paper is, however, not concerned with the relative merits of various statistical theories of fatigue, but merely with consequences of a reasonable choice from among them.

Practical. Application of the Weibull distribution received extensive attention by W. Weibull in [19], where he showed that a distribution of the general type (B1) represented certain fatigue-life data quite satisfactorily. In addition, inspection of the special "Weibull" plots accompanying the worksheets suggests that many can be fitted satisfactorily by a straight line representing a Weibull distribution, as explained below.

The manner in which these graphs are constructed is described by Weibull in [19]. A sample of Weibull-function coordinate paper used for this purpose is included in appendix A. The essence of the method is that eq (B1) may be converted, by taking logarithms twice, into

$$e(\ln L)-(e\ln a)=\ln[\ln(1/S)], \tag{B2}$$

where "ln" denotes the natural logarithm (base denoted by ϵ) and $S=S(L)$. From eq (B1) and the definitions of L_{10} and L_{50}, when $L=L_{10}$, $S(L)=.90$; and when $L=L_{50}$, $S(L)=.50$. These values substituted in eq (B2) give

[8] So named for W. Weibull (cf. [18], p. 16 ff.), who is considered to be one of the first to study it extensively.

[9] The use of a continuous instead of discrete probability distribution will introduce no appreciable error.

$$\left.\begin{aligned} e(\ln L_{10})-e(\ln a)&=\ln[\ln(1/.90)]=-2.25037\\ e(\ln L_{50})-e(\ln a)&=\ln[\ln(1/.50)]=-0.36651,\end{aligned}\right\} \tag{B2a}$$

the values on the right-hand side being obtained from [17, table 2]. These are the relationships between the parameters a, e, L_{10}, and L_{50}. The right-hand numerical values will later be denoted by $y_{.10}$, $y_{.50}$, respectively. Equation (B2) may be written

$$ex-a'=y,$$

where

$$x=\ln L,\ a'=e\ \ln a,\ y=\ln[\ln(1/S)]. \tag{B3}$$

The variables x, y correspond to the two scales shown on the Weibull-function coordinate paper in appendix A. The variable x, with unrestricted values, corresponds to the horizontal scale "Bearing life," having a logarithmic scale. The variable y is represented through the percentage surviving, S, or rather through the (vertical) scale for "bearings tested—percent"= percent failed[10]$=1-S=P$, which can vary only between 0 and 1. This scale also has nonuniform graduations, given by the iterated logarithm in (B3).

The Weibull distribution is thus seen to be equivalent to a straight-line relationship, with "Weibull slope" e, between the logarithm of fatigue life and an associated quantity y depending only on its relative rank when the fatigue lives are arranged in ascending order. Thus, goodness of fit of the straight line (B3) is equivalent to goodness of fit of a Weibull distribution to the fatigue lives L of an individual test group. In fact, one common method of statistical analysis of fatigue-life data (Freudenthal and Gumbel [6]) depends upon the use of the classical method of least squares for fitting this straight line. This method is, however, subject to certain limitations described below. Instead, an alternative method, presented in the following sections, is preferred that fits the distribution of $x=\ln L$ directly by use of order statistics.

b. Limitations of Fitting by Least Squares

In the classical method of least squares for fitting the straight-line relationship (B3) to a test group of ball-bearing data, pairs of values (x_i, y_i), $i=1, \ldots, n$, are required. The values of $x=\ln L$ are obtained from the given data. However, the variable y, measured through the percentage failing, $P=1-S$, presents difficulties. The problem of how to plot P is known as the problem of "plotting position."

It seems clear that the values, P_i, of the plotted variable, P, must somehow be related to the rank order of the bearings as they fail. A natural choice is the percentage failing: $P=f/n$, where f is the rank order of failure in a test group of n. This is not advisable for reasons discussed at length by Gumbel in [9, p. 14], where he advocates the plotting position $f/(n+1)$.[11] Other workers take different positions, and the question of plotting position must be regarded as still unsettled.

A second difficulty with the use of least squares is that as usually used it fails to take adequate account of the number of items remaining intact ("runouts") in the incompleted tests. As a final point, it is to be noted that the successive plotted points are not independent, as they represent the observed lives in increasing order. A correct use of least squares procedures would have to take into account all the intercorrelations, which is not done in the usual application of the "method of least squares." The method of order statistics described in section 3 has the advantage of avoiding the above limitations of the least squares method.

---

[10] The symbol P as used here should not be confused with the same symbol for load used in the life formula. In any event, the meaning will be clear from the context.

[11] This plotting position was also used by Weibull in [18] (cf. eq (72) and the vertical scale in figures 3 and 4 therein).

5.2. The Extreme-Value Distribution

a. Relation to Weibull Distribution

The preceding section indicates that logarithms of lives, rather than lives themselves, are the natural units in which to carry out the analysis. This idea has also been adopted by those who do not use the Weibull distribution, either because they are unaware of its existence or because they do not feel it fits their data.

If the Weibull distribution is adopted for fatigue life, L, then the variate, $x=\ln L$, has the nonnormal cumulative distribution function [12]

$$G(x)=\text{Prob}\{\ln(\text{life})\geq x\}=\text{Prob}\{\text{life}\geq \epsilon^x\}$$

$$=S(\epsilon^x)=\exp(-a^{-e}\epsilon^{ex})=\exp[-\epsilon^{(x-\ln a)/\epsilon^{-1}}], \quad -\infty<x<\infty.$$

This may be written

$$G(x)=\Phi(y)=\exp(-\epsilon^y), \tag{B5}$$

where

$$y=(x-u)/\beta, \quad -\infty<x<\infty, \tag{B6}$$

and

$$u=\ln a, \quad \beta=1/e \tag{B7}$$

are the two parameters. The distribution, $\Phi(y)$, considered as a distribution of the "reduced variable," y, has standardized parameters $u=0$, $\beta=1$, and is called the "reduced distribution."

The form (B5) is another of the three asymptotic distributions of extreme values, sometimes designated as Fisher-Tippett type I for smallest values. This distribution has been studied extensively, chiefly by E. J. Gumbel (e. g., [7, 8, 9]). In this paper, the term "extreme-value distribution" will be given to the distribution of smallest values (unless otherwise specified), although this name is frequently given to the largest-values case.

From the above discussion, it is apparent that methods pertinent to the type I extreme-value distribution (B5) are appropriate. For this purpose there is available a mathematical approach recently developed by one of the authors of this report, and described in detail in [13].

b. Characteristics

A description of the extreme-value distribution (B5), together with an interpretation of its parameters in terms of life estimates (or rather their logarithms), is essential to an understanding of the application of the method of order statistics in this paper. It will be seen that the problem of estimating life is equivalent to that of estimating the parameters u and β.

The parameters of the extreme-value distribution (B5) are depicted in figure 1 (page 291). The quantity u is the position of the mode or highest point of the (frequency) distribution. The quantity β is a scale parameter, analogous to the standard deviation, σ, in the case of the normal distribution. In fact, β is $\sqrt{6}/\pi$ (about $\frac{3}{4}$) times the standard deviation of the extreme-value distribution.

Although the two parameters, u, β, completely specify the distribution, it is very useful to introduce related quantities of the form

$$t=u+\beta y, \tag{B8}$$

which are linear combinations of parameters u and β and may thus also be regarded as parameters when known values are later assigned to y. Introduction of t makes it possible to estimate u and β simultaneously. Thus if t can be obtained as $a+by$ with a and b known and y arbitrary, then the values $u=a$, $\beta=b$ can be read off at once.

The parameter t has another highly important meaning. In figure 1 the area F under the distribution to the right of the ordinate erected at t represents the probability that a value

[12] Cf. Freudenthal and Gumbel [6], eq (2.5), (2.6), (2.8), (2.9).

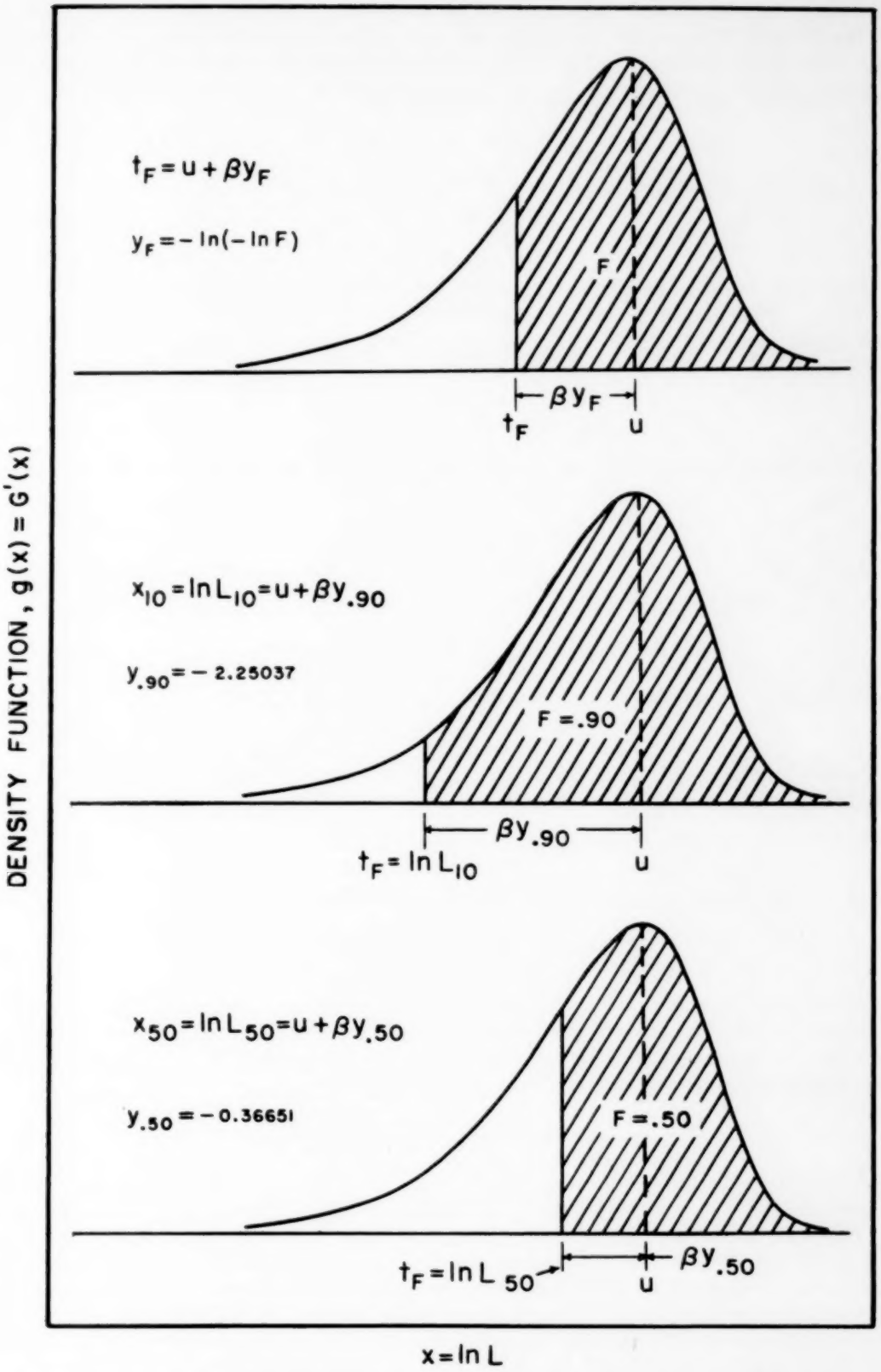


FIGURE 1. *General form of extreme-value distribution (for smallest values) showing relationship of parameters* t_F, $x_{10} = \ln L_{10}$, *and* $x_{50} = \ln L_{50}$, *to* u *and* β.

larger than t will occur. Thus t is a function of F and may be written t_F, as shown; it is designated the "upper $100F$-percentage point" of the distribution. For example, if $F=.90$, then $t=t_{.90}$ represents a value of $x=\ln L$, which will be exceeded by 90 percent of the population. This is associated with rating life L_{10} (life exceeded by 90 percent of bearings) by the relation

$$t_{.90}=x_{10}=\ln L_{10}, \tag{B9}$$

where x represents life in logarithmic units. Similarly, for median life,

$$t_{.50}=x_{50}=\ln L_{50}. \tag{B10}$$

Since the t's are regarded as parameters of the distribution, so also are x_{10} and x_{50}, and therefore L_{10} and L_{50}. These are not, of course, all independent.

In general, we have the percentage point t_F, which, expressed in terms of the original parameters u and β, may be written in the form (B8):

$$t_F=u+\beta y_F, \tag{B8a}$$

where y is a quantity depending only on the probability F, determined as follows. We have from (B8a)

$$y_F=(t_F-u)/\beta, \tag{B11}$$

i. e., y_F is the value of $(x-u)/\beta$ when x takes the value t_F. But by definition of the probability F, in view of (B5), (B6), and (B11),

$$F=\text{Prob}\{x\geq t_F\}=G(t_F)=\Phi(y_F)=\exp(-\epsilon^{y_F}). \tag{B12}$$

Thus, solving for y_F, we obtain

$$y_F=\ln(-\ln F). \tag{B13}$$

This is the reduced variable corresponding to the probability F, and may be obtained by a simple change in sign from table 2 of [17], which tabulates the function

$$-\ln(-\ln \Phi_y),$$

where Φ_y, a probability, takes on values from 0 to 1. Thus,

$$\left.\begin{aligned}&\text{for } F=.90,\ y_F=-2.25037;\\&\text{for } F=.50,\ y_F=-0.36651.\end{aligned}\right\} \tag{B14}$$

The above discussion shows that both x_{10} and x_{50} (rating and median lives in logarithmic units) may be determined once the general percentage point (B8a) is estimated by giving the two particular values (B14) to y_F.

c. Conversion From Largest to Smallest Values

The methods and numerical results developed in [13] were for problems, such as maximum gust-loads on airplanes, that required the distribution of *largest* sample values. In order to adapt this material to the distribution of *smallest* values (B5) required here, the relationships of symmetry involved in the reversal of direction must be examined with care. To avoid confusion, it is necessary to use subscripts L and S to distinguish between quantities related to the largest-value distribution from those related to the smallest-values case. No generality is lost by use of reduced variates. Thus, in (B5), x will be replaced by the reduced variate y, and, for simplicity, the symbol $G(y)$ will be used instead of $\Phi(y)$:

$$G(y)=\Phi(y)=\exp(-\epsilon^{y}). \tag{B15}$$

From this, the ("cumulative from above") distribution of smallest values is

$$\text{Prob}\{Y_S \geq y\} \equiv G_S(y) = \exp(-\epsilon^y), \qquad -\infty < y < \infty, \tag{B16}$$

where Y_S denotes the reduced smallest value. The corresponding distribution of largest values is (see Gumbel [9, eq (I), p. 21])

$$\text{Prob}\{Y_L \geq y\} \equiv H_L(y) = 1 - \text{Prob}\{Y_L < y\} = 1 - \exp(-\epsilon^{-y}) = 1 - G_S(-y), \quad -\infty < y < \infty, \tag{B17}$$

from (B16).

The corresponding relation for the density functions is obtained by differentiation, with $g_S(y) = G_S'(y)$, and $h_L(y) = H_L'(y)$:

$$g_S(y) = h_L(-y). \tag{B18}$$

Hence the two distributions are merely mirror images of each other. The moments of the distributions are related as follows:

$$\nu_{kS} \equiv E_S(y^k) = \int_{-\infty}^{\infty} y^k g_S(y) dy = \int_{-\infty}^{\infty} (-y')^k h_L(y') dy = (-1)^k \nu_{kL}. \tag{B19}$$

Thus, the means differ in sign and the variances are identical:

$$\nu_{1S} = -\gamma = -\nu_{1L}, \tag{B20}$$

$$\sigma_S^2 = \frac{\pi^2}{6} = \sigma_L^2. \tag{B21}$$

These values are given, for example, in [9, p. 23, eq (3.27)].

Finally, we need the relationships between moments of the order statistics for the two distributions. As the smallest-value distribution is a reversal of the largest-value distribution, it is natural to reverse the arrangement of the order statistics as well. This gives simpler results. Thus we are interested in the ith order statistic in the series

$$(S): \quad y_1' \geq y_2' \geq \ \ldots \ \geq y_i' \geq \ \ldots \ \geq y_n', \tag{B22}$$

where the parent distribution is that of *smallest* values. Primes will be used as a reminder that the order is *descending*, not ascending. Thus in tables B–2 and B–3 the absence of primes indicates that the order statistics are in increasing order.

(B22) is the analogue of the series

$$(L): \quad y_1 \leq y_2 \leq \ \ldots \ \leq y_i \leq \ \ldots \ \leq y_n \tag{B23}$$

of order statistics for the *largest*-value parent distribution. Whenever a distinction is necessary the subscripts S or L will be used with the y's.

From (B18) it may seem intuitively (and may be justified rigorously) that the distributions and moments of the order statistics follow the same symmetry relationships as the parent distributions, namely,

$$\left.\begin{aligned} E_S(y_i'^m) &= (-1)^m E_L(y_i^m) \\ E_S(y_i' y_j') &= E_L(y_i y_j) \\ \sigma_{S,i}'^2 &\equiv \sigma_S^2(y_i) = \sigma_L^2(y_i) \equiv \sigma_{L,i}^2 \\ \sigma_{S,ij}' &\equiv \sigma_S(y_i', y_j') = \sigma_L(y_i, y_j) \equiv \sigma_{L,ij}. \end{aligned}\right\} \tag{B24}$$

In other words, the even moments remain the same; the odd moments change only in sign.

The above development shows that the numerical results for moments of order statistics previously obtained in [13] for the largest-value case can be used here for smallest values without any substantive change.

5.3. Method of Order Statistics for Censored Samples

a. For Small Samples

Consider an independent random sample of n items from the distribution of smallest values, of which only the k smallest values can be observed. In view of the preceding discussion, it is desirable in the theoretical development to deal with the order statistics in descending order:

$$(x'_1 \geq x'_2 \geq \ldots \geq x'_{n-k}) \geq x'_{n-k+1} \geq \ldots \geq x'_n, \tag{B25}$$

where the parentheses denote the $(n-k)$ (largest) unobservable values, and the remaining k values are known. This arrangement materially simplifies the exposition. Primes will again be used to denote descending order to distinguish from ascending order, which will occur in the later parts of this section.

From the k known values it is desired to determine an estimator

$$T'_{n,k} = w'_1 x'_{n-k+1} + w'_2 x'_{n-k+2} + \ldots + w'_k x'_n, \quad k \leq n, \tag{B26}$$

(i. e., the weights w'_j) of the general parameter,

$$t_F = u + \beta y_F, \tag{B27}$$

of the extreme-value population (B5), such that T' in (B26) is (1) unbiased and (2) of minimum variance. Mathematically, this means that

$$E(T') = t_F, \tag{B28}$$

where E denotes mathematical expectation, and

$$\text{Var } (T') = \text{a minimum}, \tag{B29}$$

subject to the above condition.

From (B6),

$$x = u + \beta y, \tag{B30}$$

where y is the reduced variable and x the observed variable. From this the following relations for the order statistics x_i and y_i are apparent:

$$x'_j = u + \beta y'_j, \quad j = n-k+1, \quad n-k+2, \ldots, n, \tag{B31}$$

$$x'_{n-k+1} \geq x'_{n-k+2} \geq \ldots \geq x'_n, \tag{B32}$$

$$y'_{n-k+1} \geq y'_{n-k+2} \geq \ldots \geq y'_n, \tag{B33}$$

$$E_S(x'_j) = u + \beta E_S(y'_j). \tag{B34}$$

The values $E_S(y'_j)$ may be obtained with the aid of the table in [14]. This table gives the values of $E_L(y'_r)$ where the order statistics, y'_r, are in descending order (as indicated by the prime). The means needed in (B34) are obtained from (B24) and the evident symmetry relations

$$E_L(y_i^m) = (-1)^m E_L(y'^m_{n-i+1})$$

as

$$E_S(y'_i) = -E_L(y'_{n-i+1}). \tag{B35}$$

Reference [14] gives the values of $E_L(y'_r)$ for $r = 1(1)\min(n,26)$, $n = 1(1)10(5)60(10)100$.

From (B28) and (B31),

$$E(T') = \sum_{j=1}^{k} w'_j [u + \beta E(y'_{n-k+j})] = t_F = u + \beta y_F. \tag{B36}$$

This is required to be an identity for all values of the parameters u, β. Equating their coefficients gives the two conditions on the weights, w'_j:

$$\sum_{j=1}^{k} w'_j = 1, \quad \sum_{j=1}^{k} [E(y'_{n-k+j})]\, w'_j = y_F, \tag{B37}$$

where the numerical values $E(y'_{n-k+j})$ may be obtained from [14] as already indicated.

For the variance condition (B29), we have, in view of (B26),

$$\sigma_S^2(T') = \sum_{j=1}^{k} w'^2_j \sigma^2(x'_{n-k+j}) + \sum_{j=1}^{k} \sum_{\substack{i=1 \\ i \neq j}}^{k} w'_i w'_j \sigma(x'_{n-k+i}, x'_{n-k+j}). \tag{B38}$$

$$\mathrm{Var}(T') = [\Sigma w'^2_j \sigma'^2_{n-k+j} + \Sigma\Sigma' w'_i w'_j \sigma'_{n-k+i,\, n-k+j}]\beta^2 = V_k^{(n)'} \beta^2 \tag{B39}$$

$$= \text{minimum subject to (B28).}$$

Use of Lagrange multipliers in the same manner as in [13, pp. 50–52] gives, after differentiation, the conditions on the weights:

$$\sigma'^2_{n-k+j} w'_j + \sum_{\substack{i=1 \\ i \neq j}}^{k} \sigma'_{n-k+i,\, n-k+j} w'_j + \lambda + \mu E y'_{n-k+j} = 0, \; j = 1, 2, \ldots, k. \tag{B40}$$

For each fixed value of $k \leq n$ there are k linear equations which, with the two in (B37), form a simultaneous system of $(k+2)$ equations in the $(k+2)$ unknowns, w'_1, w'_2, . . ., w'_k, λ, μ. The values of λ and μ are useful as a check, because, if (B40) is multiplied by w_j and summed for $j = 1,2,\ldots, k$, the result is, in view of (B37) and (B39),

$$V_{k,\min}^{(n)'} + \lambda + \mu Y_F = 0. \tag{B41}$$

The minimum value, $V_{k,\min}^{(n)'}$, will be denoted by $Q'_{n,k}$.

In general, there will be a set of $(k+2)$ linear equations to solve for each $k = 2, \ldots, n$.

(1) *Case* $k = n$. For $k = n$, the matrix of coefficients and right-hand "constant terms" of (B40) and (B37) is the $(n+2)$ by $(n+3)$ matrix

$$\Lambda_n^0 = \begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \ldots & \sigma'_{1n} & 1 & Ey'_1 & 0 \\ \sigma'_{21} & \sigma'_{22} & \ldots & \sigma'_{2n} & 1 & Ey'_2 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \sigma'_{n1} & \sigma'_{n2} & \ldots & \sigma'_{nn} & 1 & Ey'_n & 0 \\ 1 & 1 & \ldots & 1 & 0 & 0 & 1 \\ Ey'_1 & Ey'_2 & \ldots & Ey'_n & 0 & 0 & y_F \end{bmatrix}. \tag{B42}$$

The ordinary $(n+2)$ by $(n+2)$ matrix of coefficients, without the constant terms, will be denoted by Λ_n. If Γ_n denotes the vector column of constant terms, then

$$\Lambda_n^0 = [\Lambda_n \vdots \Gamma_n], \tag{B43}$$

and the linear system of $(n+2)$ equations may be denoted

$$\Lambda_n W'_n = \Gamma_n, \tag{B44}$$

where W'_n denotes the column vector of the $(n+2)$ unknowns w'_1, w'_2, . . ., w'_n, λ, μ.

The coefficients of the unknowns in (B44) involve the means $E(y'_j)$, already discussed, and covariances σ'_{ij}. These values are given in table B–1 for $n=2$ to 6. The σ'_{ij} were computed by the method developed in [12]. Table B–1 also indicates how the moments for the largest values case can be obtained simply from those shown.

The $(n+2)$ solutions of (B44) are all expressible linearly in terms of the components of Γ_n. Thus the solutions all take the form

$$\left.\begin{aligned} w'_j &= a'_j + b'_j y_F, \qquad j=1,2,\ldots,n \\ \lambda &= c'_1 + d'_1 y_F \\ \mu &= c'_2 + d'_2 y_F \end{aligned}\right\}. \tag{B45}$$

Substituting these w'_j in (B39) gives an expression of the form

$$Q'_{n,n} = V^{(n)\prime}_{n,\min} = A'_n + 2B'_n y_F + C'_n y_F^2. \tag{B46}$$

The quantities a'_j, b'_j for the weights w'_j, for $n=2$ to 6, are shown in table B–2. The coefficients A'_n, B'_n, C'_n of $Q'_{n,n}$, and the values of $Q'_{n,n}$ evaluated at $F=.90$, .50, corresponding to L_{10}, L_{50}, respectively, are given in table B–3.

Calculations were limited to $n=6$ in this paper, in view of the diminishing returns in "efficiency" (see below) for increasingly larger amounts of computing. Methods suitable for larger values of n are discussed later.

Table B–3 shows that as sample size increases from $n=2$ to 6 (in the case $k=n$), the variance diminishes for the percentage-point parameters t_F for $F=.90$ and .50, i. e., $t_F=x_{10}=\ln L_{10}$ and $t_F=x_{50}=\ln L_{50}$. This is a common characteristic of the behavior of estimators for increasing sample size. Another method whereby estimators may be compared is through their efficiency.

Efficiency is a measure intended to provide a convenient standard of comparison for estimators. This is done for two estimators to be compared by dividing the variance of each into a theoretical "smallest" variance, Q_{LB}, known as the "Cramér-Rao lower bound." Further details in the case of complete samples where $k=n$, as here, may be found in [13, p. 14 and 15]; values of Q_{LB} are also indicated in this reference in table III (a).

Table B–4 shows the efficiency values so obtained, for the case $k=n$, $n=2$ to 6, as regards the order-statistics estimators for the parameters $x_{10}=\ln L_{10}$, $x_{50}=\ln L_{50}$.

These values show that for x_{10}, the efficiency, starting with under 70 percent for $n=2$, increases rapidly until 89 percent, out of a possible maximum 100 percent, is reached for $n=6$. A 90-percent-efficient estimator is generally considered to be quite good. As regards x_{50}, the efficiency is well above 95 percent for all the values of n, and for $n=6$ exceeds 99 percent. In view of results of this nature, and because of the increasingly heavy computations necessary, calculations were not carried beyond $n=6$ in [13].

The above applies to estimation of the parameters x_{10} and x_{50}, which it will be recalled are the logarithms of the actual life estimates L_{10}, L_{50}. It is believed that efficiency of the method of order statistics in obtaining estimates of actual life L_{10}, L_{50} is probably reasonable, in view of its high efficiency in estimating the logarithms, x_{10}, x_{50}.

(2) *Case* $k<n$. For the case $k<n$, the procedure is very similar. One starts with a $(k+2)$ by $(k+3)$ order matrix Λ^0_k derived from Λ^0_n in (B42) by striking out the first $(n-k)$ rows and columns. One proceeds in this manner for $k=n-1$, $n-2$, etc., until when $k=2$ the matrix becomes

$$\Lambda^0_2 = \begin{bmatrix} \sigma'_{n-1,n-1} & \sigma'_{n-1,n} & 1 & Ey'_{n-1} & 0 \\ \sigma'_{n,n-1} & \sigma'_{nn} & 1 & Ey'_n & 0 \\ 1 & 1 & 0 & 0 & 1 \\ Ey'_{n-1} & Ey'_n & 0 & 0 & y_F \end{bmatrix}, \tag{B47}$$

representing a set of 4 equations in 4 unknowns.

The resulting weights w'_j and variances $Q'_{n,k}$ were obtained in similar manner to those for $k=n$ in (B45) and (B46). These, it will be recalled, are primed quantities, associated with *descending* order of the order statistics. Because the observations, x, for successive failures naturally occur in ascending order, it is more useful for actual application, in contrast to theoretical development, to tabulate the weights and covariances for the order statistics in *ascending* order. This has been done in table B–2, giving the weights $w_i=a_i+b_i y_F$, and in table B–3, giving the variances $Q_{n,k}=A+2By_F+Cy_F^2$ for the estimators $T_{n,k}$ formed with the above weights. These variances are also evaluated for the parameters $x_{10}=\ln L_{10}$, $x_{50}=\ln L_{50}$. The relationships of these unprimed quantities to the primed ones of the previous theoretical development is merely a reversal of the order throughout, as indicated by subscripts: i. e., every a'_i is changed to the corresponding a_{k-i+1} and similarly for b'_i and w'_i. The variances Q may be shown to remain unchanged.

b. Extension to Larger Samples

Samples of more than six items are broken up into independent samples of 6 with a remainder subgroup, if necessary, of from 2 to 5 items. Because the endurance data were arranged in increasing order of life, independent random subgroups could not be obtained by simply taking groups of 6 in the (numerical) order in which they appeared on the worksheets. It was therefore first necessary to randomize the endurance lives on each data worksheet. This was accomplished by use of random numbers that were generated in the electronic computer (the SEAC) as needed.

Such artificial randomization is not desirable when it can be avoided, because the results of the calculations are then not unique, but may depend to a limited degree on the particular set of random numbers used.[13] It is therefore recommended that when the bearings in a test group are to be simultaneously run on a battery of fatigue-testing machines, the individual bearings should be recorded in advance in some more or less natural order *independent* of the order in which failure takes place in the course of the test. Natural order might be order of manufacture, order of testing, etc.

In the present investigation, each subgroup was treated as a random sample by the methods already developed for size 6 or less. That is, a "subestimator" was calculated for each subgroup and the results averaged to produce an over-all sample estimator.

An estimator, both for the individual subgroup and for the over-all sample, was obtained for each of the four population quantities:

$$u, \quad \beta, \quad t_{.90}=x_{10}=\ln L_{10}=u+y_{.90}\beta, \quad t_{.50}=x_{50}=\ln L_{50}=u+y_{.50}\beta. \tag{B48}$$

For subgroups, these four parameter estimates are given by (caret denotes "estimate of")

$$\hat{u}=\sum_{i=1}^{k} a_i x_i, \quad \hat{\beta}=\sum_{i=1}^{k} b_i x_i, \quad \hat{x}_{10}=T_{n,k}(10)=\hat{u}+y_{.90}\hat{\beta}, \quad x_{50}=T_{n,k}(50)=\hat{u}+y_{.50}\hat{\beta}, \tag{B49}$$

where $y_{.90}=-2.25037$, $y_{.50}=-0.36651$, and $x_1\leq x_2\leq \ldots \leq x_k$, $2\leq k\leq n\leq 6$, are the logarithms of the actual observed lives in a subgroup arranged in ascending order, and the a_i and b_i are read directly from table B–2. For the over-all sample estimator, the subestimators $T_{n,k}$ are merely averaged.

For later use (appendix C) the variance of the over-all estimator, T, and its relation to sample size will be considered here. Consider first the case of a *complete* sample, where no intact bearings are present because the test is run to completion. Let n be the sample size; then there are two cases, according as (1) $n\leq 6$, or (2) $n>6$.

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[13] This effect can be reduced somewhat by making a duplicate run and averaging the results, as was done in this study.

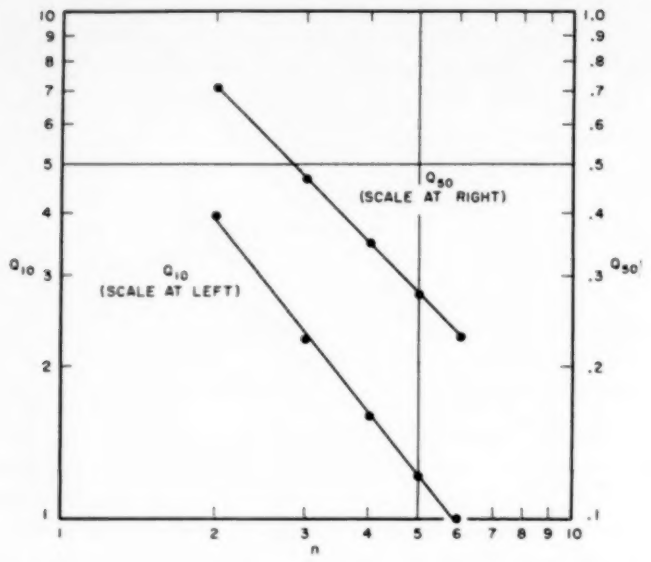

FIGURE 2. *Relationship of variances Q_{10} and Q_{50} to sample size n for n=2 to 6 (logarithmic scale in each direction).*

Q_{10} is variance of estimator of $x_{10}=\ln L_{10}$
Q_{50} is variance of estimator of $x_{50}=\ln L_{50}$
All Q's are in units of β^2

(1) $n \leq 6$. Table B–3 gives the numerical variances, Q_{10} and Q_{50}, for $n=2$ to 6. These values are plotted in figure 2 on double-logarithmic paper. The values for Q_{50} (right-hand scale) are seen to lie on a straight line of slope negative unity. This shows that at least in this case, variance is inversely proportional to sample size. For the other case, Q_{10}, a straight line also gives a reasonably good fit, and its slope appears to differ only a little from -1. Hence the underlined statement is approximately applicable here too.

(2) $n > 6$. If a sample of size $n > 6$ is broken into equal subgroups (of size 6, for example) there will generally be a remainder of size less than 6. The preceding development, when suitably modified, shows that for large n the influence of this remainder is small compared to the remaining bulk of the sample and thus the rule in question holds approximately in this case. Agreement with the rule is less close for a few cases of moderate n, but for simplicity the inverse relationship will be taken as a reasonable rule of thumb in all cases for the over-all purposes of analysis.

Two complete runs were made on the SEAC for each of the 213 test groups of data, and the two results were averaged for each group, giving values of the averages

$$\bar{u},\ \bar{\beta},\ \bar{t}_{.90}=\overline{\ln L_{10}},\ \bar{t}_{.50}=\overline{\ln L_{50}}. \tag{B50}$$

From these, the values of L_{10} and L_{50} were obtained from a table of exponentials and the Weibull slope $e=1/\bar{\beta}$ obtained as a consequence of formula (B5). An example showing the steps in calculation of L_{10}, L_{50}, and e is discussed below.

Because this investigation represents probably the most extensive mass fitting of the Weibull distribution made to date, a tabulation of the 213 values of the parameter e will be of considerable value to future applications of this distribution. This is shown in table B–6. The corresponding histogram is given in figure 3. Particular items of interest are

mean of e (all 4 companies)$=1.51$,

median e (all 4 companies)$=1.43$.

Note that 50 percent of the values are in the interval $e=1.17$ to 1.74.

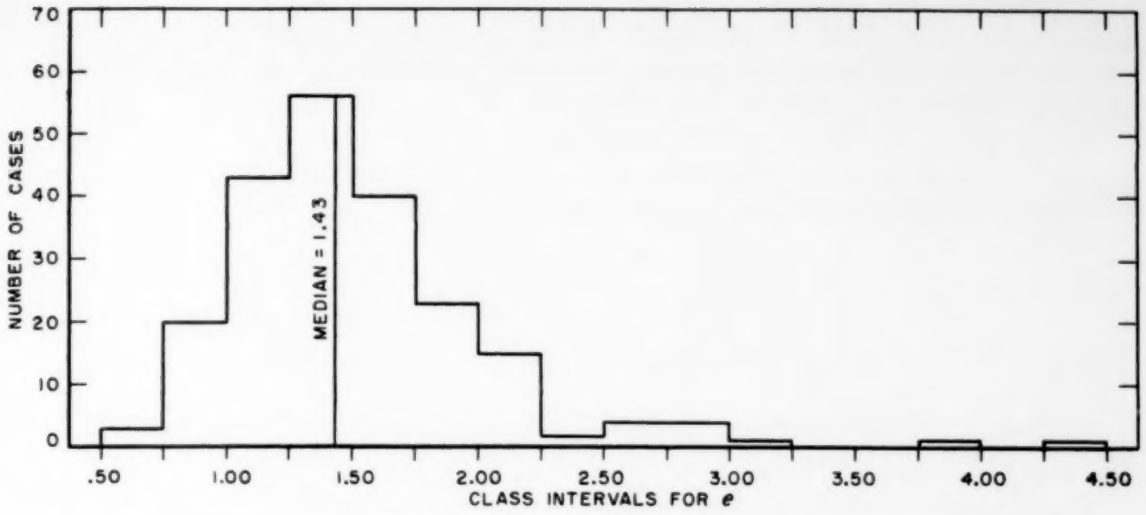

FIGURE 3. *Histogram for the 213 estimates of parameter e for companies A, B, C, and D combined.*
Data in table B–6

c. Worked Example

The example that will be given to illustrate the foregoing procedures will be the one that was worked out as a "test problem" for the SEAC before using the full set of data. The test group of bearings selected for this purpose was No. 1–1 in table A–1, for company A. The test group consisted of 24 ball bearings, of which 4 remained intact when the test was discontinued. The details of the computation for obtaining values of L_{10}, L_{50}, and Weibull slope e from the test group of data are contained in table B–5 and described in the steps below.

The endurance lives in observed (increasing) order are listed in column (1). The arrows indicate the four "run-outs," or "intacts," whose testing was discontinued at the number of million revolutions indicated. All that is known about these four bearings is that their fatigue lives exceeded the values shown.

Step 1. Randomization. The order of endurance lives in column (1) was randomized by use of a set of random numbers generated in the SEAC as part of the computation work. The result is shown in column (2) of table B–5.

Step 2. Subgroups. The lives in randomized order were divided, as shown by the lines of separation, into subgroups of size $n=6$, the maximum size for which the order-statistics weights had been computed.[14] Each subgroup was then prepared for the application of the order-statistics method by rearranging in increasing order (column (3)). Natural logarithms were then taken as in column (4).

Step 3. Weights. Each subgroup was regarded as consisting of k actual observations out of a censored sample of n. It happened here that n was 6 for every subgroup; k took the values 6, 5, 6, 3. These values are shown in the subscripts of $T_{n,k}$ written in column (3), and they determined the weights a_i and b_i to be selected from table B–2. These weights are represented in columns (5) and (6).

Step 4. Cross-products. The cross-products

$$T_{n,k}=\sum_{i=1}^{k} a_i x_i, \quad \sum_{i=1}^{k} b_i x_i$$

were then evaluated and placed as shown for each of the subgroups.

[14] As sample size 24 is an exact multiple of $n=6$, it so happened that there was no "remainder subgroup" in this case. This will not usually be true, but the procedure is identical for other values of n, differing merely in the numerical weights to be used.

Step 5. Estimates. A simple arithmetic average of the four values was taken for each of the two columns (5) and (6), and denoted by $\overline{T}_1=\hat{u}$, $\overline{T}_2=\hat{\beta}$, respectively. These are the order-statistics estimates of the two parameters u and β of the extreme-value distribution that represents the underlying Weibull distribution.

The reciprocal of $\hat{\beta}$ yields the Weibull slope $e=1.32497$. (A)

The following logarithmic life estimates were given by following linear combinations of the estimates $\hat{u}$ and $\hat{\beta}$, using the given values of $y_{.90}$, $y_{.50}$:

$$x_{10}=\ln L_{10}=\hat{u}-2.25037\ \hat{\beta}=2.982305$$

$$x_{50}=\ln L_{50}=\hat{u}-0.36651\ \hat{\beta}=4.404120$$

rating life $L_{10}=\text{antilog}\ (\hat{x}_{10})\ (\text{base}\ \epsilon)=19.2333$ million rev (B)

median life $L_{50}=\text{antilog}\ (\hat{x}_{50})\ (\text{base}\ \epsilon)=81.7872$ million rev. (C)

These three values (A), (B), and (C) represent the outcome of the calculation.

In the full-scale computing program, calculations were carried out by the SEAC to a larger number of places than is shown in the table for presentation purposes. In general, however, the number of places shown here should be adequate. The values L_{10}, L_{50}, and e shown here differ slightly from those recorded in table A–1 because the latter represent averages of two separate runs.

TABLE B–1. *Means, variances, and covariances of order statistics y_i in samples of n from the reduced extreme-value distribution $G(y)=exp(-\epsilon^{-y})$, n=2 to 6*

For distribution of *largest* values, $y_1\leq y_2\leq \ldots \leq y_n$
For distribution of *smallest* values, $y'_1\geq y'_2\geq \ldots \geq y'_n$

n	i	Means* $Es(y'_i)=-E_L(y_i)$	Variances and covariances,* $\sigma'_{ij}=\sigma'_{ji}=\sigma_{ij}=\sigma_{ji}$					
			$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
2	1	0. 11593 152	0. 68402 804	0. 48045 301				
	2	−1. 27036 285		1. 64493 407				
3	1	0. 40361 359	. 44849 796	0. 30137 144	0. 24375 810			
	2	−. 45943 263		. 65852 235	. 54629 438			
	3	−1. 67582 795			1. 64493 407			
4	1	0. 57351 263	. 34402 417	. 22455 344	0. 17903 454	0. 15388 918		
	2	−. 10608 352		. 41553 113	. 33720 966	. 29271 188		
	3	−. 81278 175			. 65180 236	. 57432 356		
	4	−1. 96351 003				1. 64493 407		
5	1	0. 69016 715	. 28486 447	. 18202 536	. 14358 737	0. 12257 865	0. 10901 329	
	2	. 10689 454		. 30849 748	. 24676 731	. 21226 644	. 18967 383	
	3	−. 42555 061			. 40598 292	. 35267 072	. 31716 095	
	4	−1. 07093 582				. 64907 319	. 58991 519	
	5	−2. 18665 358					1. 64493 407	
6	1	0. 77729 368	. 24658 20	. 15496 74	. 12121 61	. 10291 64	0. 09116 19	0. 08285 42
	2	. 25453 448		. 24854 56	. 19670 62	. 16806 28	. 14945 32	. 13619 10
	3	−. 18838 534			. 29761 59	. 25616 60	. 22887 90	. 20925 46
	4	−. 66271 588				. 40185 52	. 36145 55	. 33204 51
	5	−1. 27504 579					. 64769 96	. 59985 67
	6	−2. 36897 513						1. 64493 41

*The means are for smallest values (denoted by subscript S); for largest values, change all signs and reverse order of y's. The σ_{ij} are the same for both.

TABLE B–2. *Weights w_i for the order-statistics estimator $T_{n,k}$ for the parameter $t_p = u + \beta y_p$ of the extreme-value distribution (smallest values) from a censored sample of $n = 2$ to 6, where only the k smallest values are known*

$$T_{n,k} = w_1 x_1 + w_2 x_2 + \ldots + w_k x_k, w_i = a_i + b_i y_p$$

$$x_1 \leq x_2 \leq \ldots \leq x_k, k = 2 \text{ to } n$$

n	k		x_1	x_2	x_3	x_4	x_5	x_6
2	2	a_i	0. 0836269	0. 9163731				
		b_i	−. 7213475	. 7213475				
3	2	a_i	−. 3777001	1. 3777001				
		b_i	−. 8221012	0. 8221012				
	3	a_i	. 0879664	. 2557135	0. 6563201			
		b_i	−. 3747251	−. 2558160	. 6305411			
4	2	a_i	−. 7063194	1. 7063194				
		b_i	−. 8690149	0. 8690149				
	3	a_i	−. 0801057	. 0604316	1. 0196741			
		b_i	−. 4143997	−. 3258576	0. 7402573			
	4	a_i	. 0713800	. 1536799	. 2639426	0. 5109975		
		b_i	−. 2487965	−. 2239192	−. 0859035	. 5586192		
5	2	a_i	−. 9598627	1. 9598627				
		b_i	−. 8962840	0. 8962840				
	3	a_i	−. 2101141	−. 0860231	1. 2961372			
		b_i	−. 4343419	−. 3642463	0. 7985882			
	4	a_i	−. 0153832	. 0519642	. 1520750	. 8113440		
		b_i	−. 2730342	−. 2499429	−. 1491094	. 6720865		
	5	a_i	. 0583502	. 1088236	. 1676091	. 2462831	0. 4189341	
		b_i	−. 1844826	−. 1816564	−. 1304534	−. 0065354	. 5031278	
6	2	a_i	−1. 1655650	2. 1655650				
		b_i	−0. 9141358	0. 9141358				
	3	a_i	−. 3153968	−. 2034315	1. 5188283			
		b_i	−. 4466018	−. 3886492	0. 8352510			
	4	a_i	−. 0865378	−. 0280534	. 0649390	1. 0496521		
		b_i	−. 2858647	−. 2654739	−. 1858756	0. 7372142		
	5	a_i	. 0057311	. 0465729	. 1002523	. 1722784	. 6751653	
		b_i	−. 2015431	−. 1972753	−. 1536040	−. 0645894	. 6170118	
	6	a_i	. 0488669	. 0835221	. 1210527	. 1656192	. 2254909	0. 3554481
		b_i	−. 1458072	−. 1495332	−. 1267277	−. 0731937	. 0359868	. 4592751

TABLE B–3. *Variance $Q_{n,k}\beta^2$ of order-statistics estimator $T_{n,k}$, given in table B–2, and its numerical values $Q_{n,k}(10)=Q_{10}$, $Q_{n,k}(50)=Q_{50}$ for estimators of parameters $t_{.90}=x_{10}=\ln L_{10}$, $t_{.50}=x_{50}=\ln L_{50}$, respectively, for a censored sample of $n=2$ to 6*

Variances in units of β^2

$Q_{n,k}=A+2By_p+Cy_{p^2}$*

$x_1 \leq x_2 \leq \ldots \leq x_k$, $k=2$ to n

n	k	A	B	C	Q_{10}	Q_{50}
2	2	0. 6595467	0. 0643216	0. 7118574	3. 975015	0. 708021
3	2	. 9160386	. 4682465	. 8183654	2. 952920	. 682735
	3	. 4028637	−. 0247719	. 3447117	2. 260033	. 467327
4	2	1. 3340189	. 7720298	. 8670220	2. 250056	. 884572
	3	0. 4331573	. 1180273	. 3922328	1. 888278	. 399329
	4	. 2934587	−. 0346903	. 2252828	1. 590460	. 349150
5	2	1. 7891720	1. 0115594	. 8950462	1. 769068	1. 167910
	3	0. 5293953	0. 2353740	. 4168155	1. 580861	0. 412852
	4	. 2918142	. 0385708	. 2537913	1. 403458	. 297633
	5	. 2313953	−. 0339905	. 1666472	1. 228307	. 278697
6	2	2. 2440055	1. 2082248	. 9132926	1. 431164	1. 481035
	3	0. 6529409	0. 3332488	. 4321160	1. 341381	0. 466709
	4	. 3237185	. 1020223	. 2697162	1. 230430	. 285165
	5	. 2236063	. 0105329	. 1861069	1. 118677	. 240885
	6	. 1911738	−. 0313731	. 1319601	1. 000644	. 231897

*For $Q_{10,y_p}=-2.25037$; for $Q_{50,y_p}=-0.36651$.

TABLE B–4. *Efficiency of order-statistics estimator of logarithmic life $x_{10}=\ln L_{10}$ and $x_{50}=\ln L_{50}$ for complete samples $(k=n)$ of size $n=2$ to 6*

n	Efficiency (in percent) with respect to	
	$x_{10}=\ln L_{10}$	$x_{50}=\ln L_{50}$
2	67. 2	97. 4
3	78. 8	98. 3
4	84. 0	98. 7
5	87. 0	98. 9
6	89. 0	99. 1

TABLE B–5. *Example showing use of order statistics method of computing values of L_{10}, L_{50}, and e from endurance data*

Test group 1-1, company A

1	2	3	4	5	6
Endurance (million revolutions)			Natural logarithms x_i	Weights	
Observed order	Randomized order	Ascending order within subgroups		a_i	b_i
6. 0	119. 0	27. 5	$x_1=3.31419$	0. 048867	−0. 145807
8. 6	138. 0	69. 0	$x_2=4.23411$	. 083522	−. 149533
17. 8	146. 0	119. 0	$x_3=4.77912$	. 121053	−. 126728
18. 0	151. 0	138. 0	$x_4=4.92725$	. 165619	−. 073194
27. 5	27. 5	146. 0	$x_5=4.98361$	. 225491	. 035987
33. 5	69. 0	151. 0	$x_6=5.01728$	. 355448	. 459275
		$T_{6,6}$		$\sum_1^k a_i x_i=4.817310$	$\sum_1^k b_i x_i=0.400992$
50. 5	(150. 0)→	6. 0	$x_1=1.79176$	0. 005731	−0. 201543
51. 5	8. 6	8. 6	$x_2=2.15176$	. 046573	−. 197275
69. 0	51. 5	51. 5	$x_3=3.94158$	. 100252	−. 153604
74. 0	89. 0	89. 0	$x_4=4.48864$	. 172278	−. 064589
74. 0	109. 0	109. 0	$x_5=4.61935$	. 675165	. 617012
89. 0	6. 0	(150. 0)→	(x_6)		
		$T_{6,5}$		$\sum_1^k a_i x_i=4.446363$	$\sum_1^k b_i x_i=1.213655$
109. 0	74. 0	18. 0	$x_1=2.89037$		
118. 0	181. 0	33. 5	$x_2=3.51155$		
119. 0	141. 0	74. 0	$x_3=4.30407$	[Weights are same as for first subgroup]	
138. 0	18. 0	118. 0	$x_4=4.77068$		
141. 0	33. 5	141. 0	$x_5=4.94876$		
144. 0	144. 0	144. 0	$x_6=4.96981$		
		$T_{6,6}$		$\sum_1^k a_i x_i=4.628081$	$\sum_1^k b_i x_i=0.619440$
146. 0	17. 8	17. 8	$x_1=2.87920$	−0. 315397	−0. 446602
(150. 0)→	(153. 0)→	50. 5	$x_2=3.92197$	−. 203432	−. 388649
151. 0	(153. 0)→	74. 0	$x_3=4.30407$	1. 518828	. 835251
(153. 0)→	(153. 0)→	(153. 0)→	(x_4)		
(153. 0)→	50. 5	(153. 0)→	(x_5)		
(153. 0)→	74. 0	(153. 0)→	(x_6)		
		$T_{6,3}$		$\sum_1^k a_i x_i=4.831197$	$\sum_1^k b_i x_i=0.784853$
Over-all estimator				$\overline{T}_1=4.680738$	$\overline{T}_2=0.754735$

SUMMARY

$\overline{T}_1=\hat{u}=4.680738 \qquad \overline{T}_2=\hat{\beta}=0.754735 \qquad e=1/\hat{\beta}=1.32497$

$y_{.90}=-2.25037 \qquad y_{.50}=-0.36651$

$\hat{x}_{10}=\ln L_{10}=\hat{u}+y_{.90}\hat{\beta}=2.982305, \quad L_{10}=19.7333$

$\hat{x}_{50}=\ln L_{50}=\hat{u}+y_{.50}\hat{\beta}=4.404120, \quad L_{50}=81.7872$

TABLE B–6. *Tabulation of estimated values of Weibull slopes, e, for the 213 test groups of companies A, B, C, and D*

Weibull slope, e	Company				Total
	A	B	C	D	
0.50 to 0.74	2	1			3
.75 to .99	6	11	2	1	20
1.00 to 1.24	16	22	3	2	43
1.25 to 1.49	11	40	5		56
1.50 to 1.74	7	32	1		40
1.75 to 1.99	6	17			23
2.00 to 2.24	2	12	1		15
2.25 to 2.49		2			2
2.50 to 2.74		4			4
2.75 to 2.99		4			4
3.00 to 3.24		1			1
3.25 to 3.49					
3.50 to 3.94					
3.75 to 3.99		1			1
4.00 to 4.26					
4.25 to 4.49		1			1
Total	50	148	12	3	213
Mean	1. 33	1. 60	1. 31	1. 04	1. 51
Median	1. 27	1. 49	1. 30	1. 06	1. 43
50% interval*	1. 07 to 1. 59	1. 27 to 1. 82	1. 08 to 1. 45	0. 94 to 1. 16	1. 17 to 1. 74

*First quartile to third quartile.

6. Appendix C. Evaluation and Analysis of the Unknown Parameters in the Life Equation with Respect to Companies and Bearing Types

6.1. Summary

Equation (2a) of the main text,

$$L=\left[\frac{f_c Z^{a_1} D_a^{a_2}}{P}\right]^p,$$

expresses the dependence of fatigue-life L on the design characteristics of the deep-groove bearing (Z, D_a), the bearing load P, and the "workmanship factor" f_c. This appendix outlines the statistical methods that were used (a) to determine "best" empirical values for the parameters f_c, a_1, a_2, and p of this *life formula*, (b) to derive the associated intervals of uncertainty, and (c) to answer various questions about the values of these parameters, from the basic endurance data furnished by the ASA Subcommittee, which are summarized in appendix A. These methods are applied separately in each case to the rating life L_{10} and median life L_{50} values derived from the endurance test data as described in appendix B.

Section 6.2 summarizes the application of the statistical methods used to determine best empirical values and intervals of uncertainty for the parameters f_c, a_1, a_2, and p in the case of deep-groove bearings. Sections 6.3 to 6.6 outline the statistical analyses employed to answer various questions about the values of f_c, a_1, a_2, and p for the ball bearings of companies A, B, and C. In particular, section 6.3 gives the analysis employed to determine whether values of these four parameters are the same for the bearings of the three companies. This analysis is carried out separately for rating life L_{10} and median life L_{50}, and the postulated "complete between-companies homogeneity" is not supported in either instance. Section 6.4 gives the analysis appropriate to determining whether the data are consistent with the supposition that the value of p is the same for the three companies (regardless how the values of the other

parameters may differ); this analysis is applied to the L_{10} and L_{50} data, with an affirmative conclusion in both instances. It is concluded further that the data are consistent with the supposition that this common value for p is equal to 3.

Section 6.5 presents the analysis employed to determine whether the values of the parameters f_c, a_1, a_2, and p are the same for the three types of deep-groove bearings (B–1, B–2, and B–3) for which the data from company B were available. A negative conclusion is reached. These data are then reanalyzed to determine whether they are consistent with the supposition that $p=3$ for each of the three types, regardless of differences in the other parameters. An affirmative conclusion is reached in this case.

Section 6.6 concerns (a) the extent to which the L_{10} and L_{50} values are consistent with the supposition that the values of a_1, a_2, and p are 2/3, 1.8, and 3, respectively, as given in [15]; and (b), the determination of more precise values for f_c in those cases in which the foregoing supposition is supported.

6.2. Determination of "Best" Values for the Parameters and Their Associated Intervals of Uncertainty

As shown in section 2.2 of the main text, if natural logarithms are taken of both sides of the life equation (eq (2a)), the resulting equation expresses the logarithm of rating life L_{10} (or median life L_{50}) as a linear function of the logarithms of the characteristics of the bearing (Z,D_a), and the bearing load P, with coefficients that are simple functions of the "workmanship factor" f_c and the exponents, a_1, a_2, and p; i. e.,

$$Y=b_0+b_1x_1+b_2x_2+b_3x_3, \tag{C1}$$

where

$$\left.\begin{aligned} Y&=\ln L\\ x_1&=\ln Z\\ x_2&=\ln D_a\\ x_3&=\ln P \end{aligned}\right\} \tag{C2}$$

and

$$\left.\begin{aligned} b_0&=p\ln f_c=pa_0\\ b_1&=pa_1\\ b_2&=pa_2\\ b_3&=-p \end{aligned}\right\} \tag{C3}$$

are unknown constants to be estimated from the data.

The variables x_1, x_2, and x_3 are fixed variates. Their values are uniquely determined by the design of the bearing and the bearing loads that are used in the tests. The variable Y, on the other hand, denotes the mean values of $\ln L_{10}$, or $\ln L_{50}$, for the population of all bearings with characteristics x_1 and x_2, tested at load x_3.

In the practical situation Y is never known, but must be determined from the results of endurance tests. The methods used for obtaining such estimates of Y from endurance-test data are given in appendix B. To distinguish Y from an empirical estimate of it, the estimate will be denoted by the lower-case letter y.

Generally speaking, an estimate y is a random variable, having a probability distribution that depends on x_1, x_2, and x_3. We assume that the *mean* of this distribution is $Y=Y(x_1, x_2, x_3)$, and that its dispersion, or more precisely, its *variance*, is inversely proportional to the number of bearings w in the test group from which the estimate $y=y(x_1, x_2, x; w)$ is derived (cf.

appendix B); that is,

$$\left.\begin{aligned} &\textit{mean of } y = Y \\ &\textit{variance of } y = \frac{\sigma^2}{w}, \end{aligned}\right\} \tag{C4}$$

where σ^2 denotes some positive constant, and w is the number of bearings in the test group.

The statistical methods used to estimate the unknown parameters b_0, b_1, b_2, and b_3 from the data are termed regression techniques. The books by Anderson and Bancroft [1], Dixon and Massey [3], Hald [10], Kempthorne [11], Mood [16], and Wilks [20] give extensive discussion of these techniques. For completeness, some of the techniques and rationale of regression analysis bearing on the work embodied in this report are summarized below. More detailed discussions can be found in the above references.

Estimation. The problem of estimating the unknown parameters in the life equation can be stated as follows: Given *independent* observations $(y_\alpha; x_{1\alpha}, x_{2\alpha}, x_{3\alpha}; w_\alpha)$ from n test groups $(\alpha = 1,2, \ldots, n)$ where y_α is the estimate of ln L_{10} or ln L_{50}, $x_{1\alpha}$, $x_{2\alpha}$ are the logarithms of the bearing characteristics Z and D_a, $x_{3\alpha}$ is the logarithm of the load, and w_α is the number of individual bearings tested for the αth test group; required *to estimate* the values of the parameters b_0, b_1, b_2, and b_3, in eq (C1) using some optimum method of estimation.

Estimates for the b_i $(i=0,1,2,3)$ that are free of systematic error and have smaller variances than any other linear unbiased estimates are obtained by minimizing the quadratic form,

$$Q = \sum_{\alpha=1}^{n} w_\alpha (y_\alpha - b_0 - b_1 x_{1\alpha} - b_2 x_{2\alpha} - b_3 x_{3\alpha})^2, \tag{C5}$$

with respect to each of the b_i $(i=0,1,2,3)$.

The resulting normal equations defining the parameter estimates can be written in the form, [15]

$$\left.\begin{aligned} a_{00}\hat{b}_0 + a_{01}\hat{b}_1 + a_{02}\hat{b}_2 + a_{03}\hat{b}_3 &= g_0 \\ a_{10}\hat{b}_0 + a_{11}\hat{b}_1 + a_{12}\hat{b}_2 + a_{13}\hat{b}_3 &= g_1 \\ a_{20}\hat{b}_0 + a_{21}\hat{b}_1 + a_{22}\hat{b}_2 + a_{23}\hat{b}_3 &= g_2 \\ a_{30}\hat{b}_0 + a_{31}\hat{b}_1 + a_{32}\hat{b}_2 + a_{33}\hat{b}_3 &= g_3, \end{aligned}\right\} \tag{C6}$$

where

$$a_{ij} = \sum_{\alpha=1}^{n} w_\alpha x_{i\alpha} x_{j\alpha} \qquad i,j,=0,1,2,3. \tag{C7}$$

$$g_i = \sum_{\alpha=1}^{n} w_\alpha x_{i\alpha} y_\alpha \qquad i=0,1,2,3. \tag{C8}$$

and $x_{0\alpha}=1$ for all α.

If one defines the *residual sum of squares* by

$$S = \sum_{\alpha=1}^{n} w_\alpha \left[y_\alpha - \sum_{i=0}^{3} \hat{b}_i x_{i\alpha} \right]^2,$$

which also can be written in the alternative form

$$S = \sum_{\alpha=1}^{n} w_\alpha y_\alpha^2 - \sum_{i=0}^{3} \hat{b}_i g_i, \tag{C9}$$

then the optimum estimate of σ^2 is

$$s^2 = \frac{S}{n-4}. \tag{C10}$$

[15] A caret (^) is used here to distinguish the fact that the solutions of the normal equations are estimated values of the parameters and not the parameters themselves.

The quantity $(n-4)$ is the rank of the quadratic form S, and is termed the *degrees of freedom* associated with S.

The above method of estimation does not depend on y_α having a particular assumed probability distribution. All that is necessary to specify about the probability distribution of y_α is that it possess a finite mean and variance. The solution of the normal equations (C6) has the property that the weighted sum of squares of the deviations about y_α will be a minimum. This is a so-called "least-square" property of the solutions, but it is only a consequence of the method and is not the justification for using this method of estimation. The justification for the method is that this is the only one that results in minimum variance unbiased estimates for the b_i $(i=0, 1, 2, 3)$.

Values for b_i $(i=0, 1, 2, 3)$, and hence for a_i and p, can be obtained by using either the ln L_{10} or ln L_{50} values for y_α. In all cases where the a_i and p have been obtained for rating life L_{10}, another set of parameters have also been calculated for median life L_{50}.

Intervals of uncertainty and inferences. The methods so far discussed for finding estimates of unknown parameters need no assumption as to the form of the underlying probability distribution of y_α. However, something more must be assumed about the distribution of y_α if (a) one wishes to place an interval about an estimate of a parameter that will include the "true" (or population) value of the parameter with given assurance, or (b) if one desires to make inferences about the parameters of the life equation for the population from which the bearings are a sample.

Although the endurance lives for individual bearings may follow a Weibull distribution, the distribution of y_α will not be of this form. However, the estimate y_α (cf. appendix B) is an *average* of several independent estimates, each based on linear functions of six or less order statistics. Hence, by the central limit theorem, the distribution of the estimate y_α will be approximated by a normal distribution when n is large (cf. Cramér [2, p. 213]). The statistical tests of significance used in this report are not greatly affected by moderate departures from normality. Therefore, for making all inferences, it will be further assumed that the estimates y_α follow a normal distribution.

The intervals of uncertainty calculated for each parameter are 95-percent confidence limits (equivalent to the usual "two-sigma" limits), which were referred to in the main text. Confidence limits for the parameter $p=-b_3$ can be calculated by using conventional methods. However the confidence limits for a_0, a_1, and a_2 are somewhat more complicated. The method used here, for this purpose, is sometimes referred to as Fieller's theorem (cf. Fieller [4]).

In order to make inferences about the parameters with respect to the different companies or bearing types, certain statistical tests of significance were used in this paper. These are all based on a test statistic F, termed the variance ratio or F-ratio, which takes the form

$$F=\frac{Q_1/\nu_1}{Q_2/\nu_2},$$

where Q_1 and Q_2 are quadratic forms calculated from the data and ν_1, ν_2 are the respective ranks of the quadratic forms. The explicit expressions for Q_1 and Q_2 depend upon the particular hypothesis being tested. The subsequent sections that employ a variance-ratio statistic also give the explicit expressions for the two quadratic forms.

If the hypothesis being tested is true, then the calculated variance ratio will deviate from unity in accordance with its tabulated distribution. However, if the hypothesis is false, then the variance ratio will be substantially greater than unity, and the "more false" the stated hypothesis, the larger the value for the variance ratio. Thus when the hypothesis tested is false, this will be detected by an abnormally large F-ratio. In order to objectively determine whether a calculated variance ratio is *significantly* greater than unity, one selects from tables of the variance-ratio distribution a *critical value* of F, such that there is only a small probability of the calculated variance ratio exceeding the critical value from purely chance causes. The critical value for F used for all variance-ratio tests, in this paper, has been selected so that there

is only a probability of .05 of its being exceeded by a calculated variance ratio from purely chance causes. This critical value will be denoted by $F_{.05}(\nu_1,\nu_2)$. More extensive discussions of these procedures are given in Kempthorne [11, chap. 5].

All statistical tests of significance to test relevant hypotheses have been carried out both for rating life L_{10} and median life L_{50}.

6.3. Analysis to Determine Whether Companies Have Common Values for All the Parameters in the Life Equation

This section deals with the details of estimating the values of the parameters in the life equation for each company. Furthermore, a statistical analysis is made to determine whether the companies have common values for all the parameters in the life equation. The F-ratio (eq (C16)), which is used to test this hypothesis, is obtained from the following procedures: A single set of parameters, b_i $(i=0, 1,2,3)$, is obtained by first fitting all the data, irrespective of company, to the logarithmic life equation, and then calculating the resulting residual sums of squares S (eq (C14)) having 206 degrees of freedom. If the hypothesis of common values for all the parameters is not true, then a better fit to the data can be made by fitting the life equation separately to each company. These calculations result in the individual residual sums of squares S_1, S_2, and S_3 (eq (C13)) having 46, 144, and 8 degrees of freedom, respectively. Thus the total residual sums of squares $(S_1+S_2+S_3)$ will have $46+144+8=198$ degrees of freedom. Then the difference between S and $(S_1+S_2+S_3)$, i. e., $\{S-(S_1+S_2+S_3)\}$, is also a quadratic form having $206-198=8$ degrees of freedom. If a substantially better fit was obtained by fitting a separate life equation to the data for each company, as compared to a single life equation, the difference between the two residual sums of squares $\{S-(S_1+S_2+S_3)\}$ will be large. To determine whether this difference is statistically significant the variance ratio (C16) is employed.

Mathematical formulation. It will be convenient to adopt the following notation: Let the superscript $(u)=1, 2, 3$ refer to companies A, B, and C, respectively. Also, for each company, let $\hat{b}_i^{(u)}$ $(i=0,1,2,3)$ refer to the estimates of the parameters in eq (C1), and $a_{ij}^{(u)}$, $g_i^{(u)}$ denote the sums of cross products defined in eq (C7) and (C8)[16]. Then the normal equations that give the parameter estimates for the uth company are

$$\sum_{j=0}^{3} a_{ij}^{(u)} \hat{b}_j^{(u)} = g_i^{(u)} \qquad i=0,1,2,3, \tag{C11}$$

and the estimates for the parameters a_i $(i=0, 1, 2)$ and p, in the life equation, are obtained from the relations given by eq (C3). These results are summarized in tables 6 and 3, respectively, in the main text.

The hypothesis that all parameters in the life equation are the same for each company is equivalent to the hypothesis that

$$b_i^{(1)}=b_i^{(2)}=b_i^{(3)} \qquad i=0,1,2,3. \tag{C12}$$

Define the residual sum of squares for the uth company by

$$S_u=\sum_{\alpha=1}^{n_u} w_{u\alpha} y_{u\alpha}^2 - \sum_{i=0}^{3} \hat{b}_i^{(u)} g_i^{(u)} \qquad u=1,2,3, \tag{C13}$$

and let

$$S=\sum_{u=1}^{3} \sum_{\alpha=1}^{n_u} w_{u\alpha} y_{u\alpha}^2 - \sum_{i=0}^{3} \hat{b}_i g_i, \qquad g_i=\sum_{u=1}^{3} g_i^{(u)}, \tag{C14}$$

where b_i $(i=0,1,2,3)$ are the estimates obtained from the solutions of the normal equations

[16] As it is only the ratio of the numbers of observations that is important for weighting, the weights w used in the calculation of the sums of cross products in all the analyses have been taken as integral multiples of 5; e. g., if the number of bearings in a test group is 26 (say), then $w=5$.

without regard to company differences, i. e.,

$$\sum_{j=0}^{3}\left[\sum_{u=1}^{3} a_{ij}^{(u)}\right]\hat{b}_j=\sum_{u=1}^{3} g_i^{(u)} \qquad i=0,1,2,3. \tag{C15}$$

Then the variance ratio for testing the hypothesis given by eq (C15) is

$$F=\frac{(S-S_1-S_2-S_3)/8}{(S_1+S_2+S_3)/198}, \tag{C16}$$

and the critical F value is $F_{.05}$ (8, 198)=1.98.

Table C–1 summarizes the values for S_u (u=1,2,3) and the other quantities needed to evaluate eq (C16).

TABLE C–1. *Synopsis of calculations to determine whether companies have common values for all parameters in life equation*

	Degrees of freedom	L_{10}		L_{50}	
		S_u	Mean square	S_u	Mean square
Companies combined	206	446. 684633	------	348. 285324	------
A	46	102. 552212	------	89. 734079	------
B	144	285. 121962	------	201. 935797	------
C	8	6. 160858	------	8. 357822	------
Sum	198	393. 835032	1. 9891	300. 027698	1. 5153
Difference	8	52. 849601	6. 6062	48. 257626	6. 0322
From eq (C16)		$F=\frac{6.6062}{1.9891}=3.32$		$F=\frac{6.0322}{1.5153}=3.98$	

The values of the F-ratio (eq (C16)) calculated from the quantities in table C–1 are

$$L_{10}: \quad F=3.32$$

$$L_{50}: \quad F=3.98$$

These calculated F values are both larger than the critical value, $F_{.05}$ (8, 198)=1.98; actually the probability is less than .0005 of having an F-ratio as large as those above from purely chance causes. Thus from the above statistical tests of significance, the conclusion is reached that *the three companies do not have common values for all of the parameters in the life equation.*

6.4. Analysis to Determine Whether Companies Have a Common Value for the Exponent p

The previous analysis resulted in the conclusion that the parameters in the life equation are different for each company. However, this does not exclude the possibility that all companies may have a common p, even though the a_i (i=0,1,2) differ from company to company. This section discusses the analysis made to determine whether the companies have a common value for the exponent p. The analysis given here consisted of the following procedure. First, the logarithmic life equation (C17), having a common value of p, but allowing the a_i to vary for each company, was fitted to all the data, and the resulting residual sum of squares S' (eq (C20)), having 200 degrees of freedom, was calculated. The total residual sum of squares from fitting the life equation separately to each company (allowing p to vary in addition to the a_i) is given by $(S_1+S_2+S_3)$ having 198 degrees of freedom (cf. section 6.3 of this appendix).

Then the reduction in the residual sum of squares achieved by using a different exponent p for each company is $\{S'-(S_1+S_2+S_3)\}$ having $200-198=2$ degrees of freedom. To test whether this reduction in the residual sums of squares is statistically significant, the variance ratio (C21) is employed.

Mathematical formulation. The logarithmic life equation, having a common value for the exponent p, can be written for the αth test group in the uth company as

$$Y_{u\alpha}=b_0^{(u)}+b_1^{(u)}x_{1\alpha}^{(u)}+b_2^{(u)}x_{2\alpha}^{(u)}+b_3x_{3\alpha}^{(u)} \qquad \alpha=1, 2, \ldots, n_u;\ u=1, 2, 3. \tag{C17}$$

Note that although each company has the same parameter b_3 in the above equation, the parameters $b_0^{(u)}$, $b_1^{(u)}$, and $b_2^{(u)}$ are different for each of the three companies. Thus there are 10 different parameters, i. e., $b_i^{(u)}$ $(i=0,1,2;\ u=1,2,3)$ and b_3, to be estimated from the data.

The normal equations for estimating these parameters are

$$\left.\begin{aligned} &\sum_{j=0}^{2} a_{ij}^{(u)}\hat{b}_j^{(u)}+a_{i3}^{(u)}\hat{b}_3=g_i^{(u)} \qquad i=0, 1, 2;\ u=1, 2, 3 \\ &\sum_{u=1}^{3}\sum_{j=0}^{2} a_{3j}^{(u)}\hat{b}_j^{(u)}+a_{33}\hat{b}_3=g_3, \end{aligned}\right\} \tag{C18}$$

where

$$a_{33}=\sum_{u=1}^{3} a_{33}^{(u)} \qquad g_3=\sum_{u=1}^{3} g_3^{(u)}.$$

Thus the set of equations given by (C18) is a system of 10 linear equations in 10 unknowns. Once the solutions are obtained, the estimates for $a_i(i=0,1,2)$ in the life equation are calculated from the relationship

$$\hat{a}_i^{(u)}=-\frac{\hat{b}_i^{(u)}}{\hat{b}_3} \qquad i=0, 1, 2;\ u=1, 2, 3. \tag{C19}$$

The residual sums of squares (denoted by S') associated with fitting the life formula (C17) to the data is given by

$$S'=\sum_{u=1}^{3}\sum_{\alpha=1}^{n_u} w_{u\alpha}y_{u\alpha}^2-\sum_{u=1}^{3}\sum_{i=0}^{2}\hat{b}_i^{(u)}\ g_i^{(u)}-\hat{b}_3g_3. \tag{C20}$$

Then, to test the hypothesis that the companies have a common value of p, regardless of the values for the other parameters in the life equation, the variance ratio

$$F=\frac{(S'-S_1-S_2-S_3)/2}{(S_1+S_2+S_3)/198}, \tag{C21}$$

having 2 and 198 degrees of freedom is used. The critical F value is $F_{.05}(2, 198)=3.04$.

The values of S' for both L_{10} and L_{50} are

$$\left.\begin{aligned} &L_{10}: \quad S'=393.272847 \qquad \text{d. f.}=200 \\ &L_{50}: \quad S'=301.687871 \qquad \text{d. f.}=200, \end{aligned}\right\} \tag{C22}$$

and the calculated [17] F values (using eq (C21)) give

$$\left.\begin{aligned} &L_{10}: \quad F=-0.141 \\ &L_{50}: \quad F=0.548. \end{aligned}\right\} \tag{C23}$$

---

[17] From theory, the calculated value for the F-ratios can never be negative. The reason for the negative value of F for L_{10} is that the value for the numerator of eq (C21) is only accurate numerically to one decimal place on account of round-off errors arising from the solution of the normal equations (C18). Thus, if the hypothesis of a common p value is true, then the F-ratio will *not* be large and round-off errors may affect the resulting calculation. Alternatively, if the null hypothesis is false, then the calculated F-ratio will be larger then 3.04 and the round-off error should be of no consequence.

Because both variance ratios are smaller than the critical value $F_{.05}$ (2, 198)=3.04, the conclusion drawn from this statistical analysis is that the data support the hypothesis of a common value of p for the three companies. This holds both for rating life (L_{10}) and median life (L_{50}). The values for the common p are given in table 2 of the main text. The values for the remaining constants in the life equation $a_i(i=0,1,2)$ are found from the relations (C19). These results are summarized in table 5 of the main text.

6.5. Analysis to Determine Whether the Three Bearing Types From Company B Have Consistent Parameter Values

The analyses, discussed in previous sections, dealt with determining whether there are differences in the parameters of the life equation between companies. This section investigates (a) whether three different bearing types made by company B have common values for all the parameters in the life equation, and (b) whether the exponents p calculated for each bearing type are consistent with the value of $p=3$.

The analysis for (a) is similar to the analysis made in section 6.3 of this appendix; i. e., separate life equations were fitted to each bearing type and the resulting residual sum of squares was compared with the residual sum of squares arising from fitting a single equation to all data from company B, irrespective of bearing type. The variance ratio for statistically testing (a) is given by eq (C27).

The analysis for (b) was governed by the following considerations. If the true (or population) value of the exponent p is $p=3$, regardless of bearing type, then the estimates for p obtained by fitting a separate life equation to each bearing type should not differ from $p=3$ by more than the dispersion inherent in the endurance lives of the bearings. The agreement of the values of p estimated for each bearing type with $p=3$ is tested for statistical significance by the variance ratio (C28).

Mathematical formulation. The 148 test groups from company B can be divided into three bearing types corresponding to 37 groups for B–1, 94 groups for B–2, and 17 groups for B–3–type bearings. Let these types be denoted by $v=1, 2, 3$, respectively. Also define

$$\left.\begin{aligned} A_{ij}^{(v)} &= \sum_{\alpha=1}^{n_v} w_{v\alpha}\, x_{i\alpha}^{(v)} x_{j\alpha}^{(v)} \qquad i,j=0,1,2,3;\ v=1,2,3 \\ G_i^{(v)} &= \sum_{\alpha=1}^{n_v} w_{v\alpha}\, x_{i\alpha}^{(v)} y_{v\alpha}^{(v)} \qquad i=0,1,2,3;\ v=1,2,3, \end{aligned}\right\} \tag{C24}$$

where n_v is the number of test groups for bearing type v. Then the normal equations, which determine the estimates for the parameters in the logarithmic life equation, are

$$\sum_{j=0}^{3} A_{ij}^{(v)} \hat{b}_j^{(v)} = G_i^{(v)} \qquad i=0,1,2,3;\ v=1,2,3. \tag{C25}$$

Thus for each bearing type, the values for the parameters $a_i(i=0,1,2)$ and p in the life equation can be estimated from the relations

$$\hat{a}_i^{(v)} = -\frac{\hat{b}_i^{(v)}}{\hat{b}_3^{(v)}},$$

$$\hat{p}^{(v)} = -\hat{b}_3^{(v)}.$$

Therefore the residual sum of squares for the vth bearing type is

$$S^{(v)}=\sum_{a=1}^{n_v} w_{va}y_{va}^2-\sum_{i=0}^{3}\hat{b}_i^{(v)}G_i^{(v)} \qquad v=1,\ 2,\ 3, \tag{C26}$$

having (n_v-4) degrees of freedom. Then the hypothesis of common parameters for the three bearing types can be tested by the variance ratio,

$$F=\frac{(S_2-S^{(1)}-S^{(2)}-S^{(3)})/8}{(S^{(1)}+S^{(2)}+S^{(3)})/136}, \tag{C27}$$

having a critical value of $F_{.05}(8,\ 136)=2.01$.

Since the analysis given in section 6.4 of this appendix reached the conclusion that all companies have a common value for p, and since this value (cf. table 2 of main text) with its associated uncertainty includes the value $p=3$ given in [15], it seems also desirable to test a second hypothesis that the value of p for each bearing type is consistent with $p=3$. The F-ratio for this hypothesis is given by

$$F=\frac{\frac{1}{3}\left[\frac{(\hat{p}^{(1)}-3)^2}{C_{33}^{(1)}}+\frac{(\hat{p}^{(2)}-3)^2}{C_{33}^{(1)}}+\frac{(\hat{p}^{(3)}-3)^2}{C_{33}^{(1)}}\right]}{(S^{(1)}+S^{(2)}+S^{(3)})/136}, \tag{C28}$$

where the $p^{(v)}(v=1,2,3)$ refer to the estimates of p obtained for each bearing type, and $C_{33}^{(v)}$ $(v=1,2,3)$ is the element occurring in the last row, last column of the inverse matrix to $\|A_{ij}^{(v)}\|$ $(v=1,2,3)$. The critical value for the variance ratio (C28) is $F_{.05}\ (3,\ 136)=2.67$.

The estimates of $a_i^{(v)}$ $(i=0,\ 1,\ 2)$ and $p^{(v)}$ obtained from the solutions of the normal equations (C25) are summarized in tables 7 and 4, respectively, in the main text. The calculations for the variance ratio (C27) are summarized in table C-2.

Corresponding to the hypothesis that the three bearing types have the same parameters in the life equation, the calculated variance ratios (C27) yield

$$\left.\begin{aligned} L_{10}\text{:}\quad & F=2.05 \\ L_{50}\text{:}\quad & F=2.88. \end{aligned}\right\} \tag{C29}$$

TABLE C–2.—*Synopsis of calculations to determine whether all bearing types have common parameters in life equation (company B only)*

Type of bearing	Degrees of freedom	L_{10}		L_{50}	
		$S^{(v)}$	Mean square	$S^{(v)}$	Mean square
All types combined	*144	*285. 121962		*201. 935797	
B–1	33	68. 558298		33. 016973	
B–2	90	174. 325939		131. 142017	
B–3	13	11. 537480		8. 505638	
Sum	136	254. 421716	1. 8707	172. 664628	1. 2696
Difference	8	30. 700246	3. 8375	29. 271169	3. 6589
From eq (C27)		$F=\frac{3.8375}{1.8707}=2.05$		$F=\frac{3.6589}{1.2696}=2.88$	

*From table C–1.

Because the critical F value is $F_{.05}$ (8, 136)=2.01, both the L_{10} and L_{50} calculated variance ratios are statistically significant. Therefore, one could conclude from the above F-ratios that the *data support the hypothesis that parameters do differ between bearing types.* However, this *does not* exclude the possibility that the values for the exponent p are consistent with the value $p=3$. Substituting the appropriate quantities in eq (C28) results in

$$\left.\begin{aligned} L_{10}\text{:}\quad & F=1.18 \\ L_{50}\text{:}\quad & F=2.32 \end{aligned}\right\}, \tag{C30}$$

where

$$C_{33}^{(1)}=0.53294$$

$$C_{33}^{(2)}=0.107395$$

$$C_{33}^{(3)}=0.396744.$$

Thus, as both of the above calculated F-ratios are less than the critical value, $F_{.05}$ (3,136) =2.67, the conclusion can be made that *the values of p are consistent with $p=3$ for different bearing types*, although possibly some (or all) of the other parameters in the life formula (i. e., $a_i(i=0,1,2)$ may differ among the different bearing types.

The values for $\hat{a}_i^{(v)}$ $(i=0,1,2;\ v=1,2,3)$ arising from the analysis by ball-bearing types have very large confidence limits (intervals of uncertainty). This is mainly due to the fact that an analysis restricted to one bearing type is essentially an analysis on bearings having almost the same values for Z and D_a. In order to estimate the $a_i^{(v)}$ with good precision, it is necessary to have results for bearings having wide variations with respect to Z and D_a. Thus the estimates for a_i based on all bearing types for company B (table 5 or table 6) have substantially smaller confidence intervals as compared to the intervals based only on a single bearing type.

6.6. Determination and Analysis of f_c Based on the Parameter Values $a_1=2/3$, $a_2=1.8$, and $p=3$

The values for the parameters a_1, a_2, and p given in [15] are $a_1=2/3$, $a_2=1.8$, and $p=3$. If these parameter values are valid for the data at hand, then more precise estimates for the "workmanship" parameter a_0 (or f_c) can be made for each company or bearing type. These will generally have better precision compared to the estimates of a_0 made when the other parameters in the life equation are simultaneously estimated along with a_0. This section considers the problem of verifying whether the parameter values, given above, are valid for the given data, and for those cases where this is true, estimates of a_0 (or f_c) are obtained assuming these values for the other parameters.

The procedure for determining whether the values $a_1=2/3$, $a_2=1.8$, and $p=3$ are valid for a given classification of the data (with respect to a company or bearing type) is to fit the data to the life equation, using the assumed values for a_1, a_2, and p. Thus there is only one unknown parameter, a_0, in the life equation to be estimated. Then the resultant residual sum of squares, denoted by R (eq (C37)), can be calculated having $(n-1)$ degrees of freedom. Alternatively, the life equation can be fitted to the data such that *all* the unknown parameters are simultaneously estimated. The residual sum of squares from this latter fit, S, will have $(n-4)$ degrees of freedom. Then, if the above parameter values are not consistent with the given data, R will be appreciably larger than S. The variance ratio (C38) is used to determine if the difference between these two residual sums of squares, i. e., $(R-S)$ having $[(n-1)-(n-4)]=3$ degrees of freedom, is statistically significant.

Mathematical formulation. Let n be the number of test groups within a particular classification (either by company or bearing type). Then, assuming the values $a_1=2/3$, $a_2=1.8$, and

$p=3$, the logarithmic life equation for the αth test group can be written as

$$Y_\alpha=3\{a_0+x_\alpha\} \qquad \alpha=1, 2, \ldots, n, \tag{C31}$$

where

$$x_\alpha=\left[\frac{2}{3}\,x_{1\alpha}+1.8\,x_{2\alpha}-x_{3\alpha}\right]. \tag{C32}$$

The resulting normal equation for estimating a_0 is

$$\hat{a}_0=\frac{\sum_{\alpha=1}^{n}\frac{w_\alpha y_\alpha}{3}-\sum_{\alpha=1}^{n}w_\alpha x_\alpha}{\sum_{\alpha=1}^{n}w_\alpha}, \tag{C33}$$

which also can be written as a function of the sums of cross products g_i, a_{ij}, e. g.,

$$\hat{a}_0=\frac{1}{a_{00}}\left\{\frac{g_0}{3}-\left(\frac{2}{3}\,a_{01}+1.8\,a_{02}-a_{03}\right)\right\}. \tag{C34}$$

Hence the estimated variance of $\hat{a}_0$ is

$$\text{variance }(\hat{a}_0)=\frac{s^2}{9a_{00}}, \tag{C35}$$

where

$$s^2=\frac{R}{n-1}=\frac{9\left[\sum_{\alpha=1}^{n}w_\alpha\left(\frac{y_\alpha}{3}-x_\alpha\right)^2-a_{00}\hat{a}_0^2\right]}{n-1}. \tag{C36}$$

The residual sum of squares R, having $(n-1)$ degrees of freedom, can also be written as a function of the sums of cross products,

$$R=9\left\{\frac{1}{9}\sum_{\alpha=1}^{n}w_\alpha y_\alpha^2-\frac{4}{9}\,g_1-1.2\,g_2+\frac{2}{3}\,g_3+\frac{4}{9}\,a_{11}+3.24\,a_{22}+a_{33}+2.4\,a_{12}-\frac{4}{3}\,a_{13}-3.6\,a_{23}-a_{00}\hat{a}_0^2\right\}. \tag{C37}$$

In the analyses made in the preceding sections, the ball-bearing data have been analyzed with respect to individual companies or bearing types. It thus seems desirable to determine whether the data within these classifications support the hypothesis that $a_1=2/3$, $a_2=1.8$, and $p=3$. The variance ratio used to test this hypothesis is

$$F=\frac{(R-S)/3}{S/(n-4)}, \tag{C38}$$

where, for testing within companies,

$$S=\left\{\begin{array}{l}S_1 \text{ for company A}\\ S_2 \text{ for company B}\\ S_3 \text{ for company C}\end{array}\right\} \quad \text{defined by eq (C13) and given in table C–1,}$$

and for testing within bearing types, $a_{ij}^{(w)}$ is replaced by $A_{ij}^{(r)}$,

$$S=\left\{\begin{array}{l}S^{(1)} \text{ for B–1}\\ S^{(2)} \text{ for B–2}\\ S^{(3)} \text{ for B–3}\end{array}\right\} \quad \text{defined by eq (C26) and given in table C–2,}$$

and R (eq (C37)) refers to the calculated residual sum of squares within the particular classification.

Table C–3 summarizes with respect to companies A, B, and C the values of $\hat{a}_0$, R, the calculated F-ratio (eq C38), and the critical F value. Table C–4 summarizes the same quantities for the B–1, B–2, and B–3-type bearings made by company B.

The results of this analysis show that the values $a_1=2/3$, $a_2=1.8$, and $p=3$ are consistent for rating life L_{10} data with respect to each of the three companies. However, for median life L_{50}, these assumed parameter values are consistent only for the data from company A.

The results for the same analysis made on the three different bearing types, indicate that the assumed parameter values are consistent only for the B–1 and B–2-type bearings with respect to median life L_{50}.

It is interesting to note that the analysis for company B (ignoring bearing types) showed that the assumed values for the parameters are consistent with the L_{10} data. However, a finer analysis by bearing type revealed that these values are not valid for the B–3-type bearings. This apparent inconsistency stems from the fact that the analysis for company B, taken as a whole, is dominated by those bearing types having the larger number of test groups, i. e., B–1 and B–2, and for these types the parameter values were found to be consistent with the L_{10} data.

The estimates for f_c, assuming $a_1=2/3$, $a_2=1.8$, and $p=3$, are summarized in table 8 of the main text for rating life L_{10}. This summary also includes the value of f_c for company D computed from only three test groups. (Because of the small number of test groups, it was not possible to verify whether the assumed parameter values are valid for these data.)

TABLE C–3. *Summary of computations (by companies) to test hypothesis that data are consistent with assumed values $a_1=2/3$, $a_2=1.8$, $p=3$*

Company	L_{10}			L_{50}			Critical F
	$\hat{a}_0=\ln f_c$	R	F	$\hat{a}_0=\ln f_c$	R	F	
A	8. 4205	102. 539286	*−0. 06	8. 9382	94. 367916	0. 79	2. 80
B	8. 5021	287. 613198	0. 42	8. 9254	217. 560672	3. 71	2. 65
C	8. 1001	8. 375562	0. 96	8. 5832	8, 488584	9. 92	3. 59

*Negative value due to rounding in calculations.

TABLE C–4. *Summary of computations (by bearing type) for company B to test hypothesis that data are consistent with assumed values $a_1=2/3$, $a_2=1.8$, $p=3$*

Type	L_{10}			L_{50}			Critical F
	$\hat{a}_0=\ln f_c$	R	F	$\hat{a}_0=\ln f_c$	R	F	
B–1	8. 4575	73. 821978	0. 84	8. 8636	44. 883126	3. 95	2. 86
B–2	8. 5236	181. 128312	1. 17	8. 9482	141. 731604	2. 42	2. 70
B–3	8. 5203	27. 014247	5. 81	9. 0022	18. 905517	5. 21	3. 24

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Journal of Research of the National Bureau of Standards Vol. 57, No. 5 November 1956

Contents

		Page
Efficiency of 4π-crystal-scintillation counting: 1. Experimental technique and results. C. C. Smith, H. H. Seliger, and J. Steyn.	RP2716	251
Efficiency of 4π-crystal-scintillation counting: 2. Dead-time and coincidence corrections. W. B. Mann and H. H. Seliger.	RP2717	257
Entropy changes in rarefaction waves. Robert F. Dressler.	RP2718	265
Statistical investigation of the fatigue life of deep-groove ball bearings. J. Lieblein and M. Zelen.	RP2719	273

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